

Deformations of Generalized Kähler Structures and Bihermitian Structures

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Abstract

Let (X, J) be a compact Kähler manifold with a non-zero holomorphic Poisson structure β . If the obstruction space for deformations of generalized complex structures on (X, J) vanishes, we obtain a family of deformations of non-trivial bihermitian structures (J, J_t^-, h_t) on X by using β . In addition, if the class $[\beta \cdot \omega]$ does not vanish for a Kähler form ω , then the complex structure J_t^- is not equivalent to J for small $t \neq 0$ under diffeomorphisms. Our method is based on the construction of generalized complex and generalized Kähler structures developed in [10] and [11]. As applications, we obtain such deformations of bihermitian structures on del Pezzo surfaces, the Hirtzebruch surfaces F_2, F_3 and degenerate del Pezzo surfaces. Further we show that del Pezzo surfaces S_n ($5 \leq n \leq 8$), F_2 and degenerate del Pezzo surfaces admit bihermitian structures for which (X, J_t^-) is not biholomorphic to (X, J) for small $t \neq 0$.

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Introduction

A bihermitian structure on a C^∞ manifold X consists of a pair of integrable complex structures J^+ and J^- with a Riemannian metric h which is hermitian with respect to both J^+ and J^- . If a complex manifold (X, J) has a bihermitian structure (J^+, J^-, h) with the property $J^+ = J$, then we say that (X, J) admits a (compatible) bihermitian structure. A bihermitian structure (J^+, J^-, h) is *distinct* if the complex manifold (X, J^+) is not biholomorphic to (X, J^-) . We have the two $\bar{\partial}$ -operators $\bar{\partial}_+$ and $\bar{\partial}_-$ corresponding to the complex structures J^+ and J^- respectively. In this paper we always assume that a bihermitian structure satisfies the condition,

$$-d_+^c \omega_+ = d_-^c \omega_- = db, \quad (0.1)$$

where $d_\pm^c = \sqrt{-1}(\bar{\partial}_\pm - \partial_\pm)$ and ω_\pm denote the fundamental 2-forms with respect to J^\pm and b is a real 2-form. (Note that if $H := -d_+^c \omega_+ = d_-^c \omega_-$ is not d -exact but d -closed, (J^+, J^-, h) is called the H -twisted bihermitian structure.) There is a research of compact complex surfaces which admit bihermitian structures from the view point of Riemannian geometry [2]. Bihermitian structures with the condition (0.1) appeared on the target space of $(2, 2)$ supersymmetric sigma model [7]. Surprisingly it turned out that there is a one to one correspondence between generalized Kähler structures and bihermitian structures with the condition (0.1)[12]. It is thus expected that the construction of interesting and various generalized Kähler structures would be a major step of development of the theory of bihermitian structures. Let (X, J) be a compact Kähler manifold with a Kähler form

ω . In the paper [10, 11], the author constructed a family of deformations of bihermitian structures by using a holomorphic Poisson structure β . In the present paper, we shall obtain another family of deformations of bihermitian structures (J_t^+, J_t^-, h_t) of (X, J) , starting with the ordinary Kähler structure which satisfies $J_t^+ = J$ for all t , $J_0^- = J$ and $J_t^- \neq \pm J$ for small $t \neq 0$, where t is a parameter of deformations.

Throughout this paper we will assume that X is the underlying differential manifold of a complex manifold $M = (X, J)$ with the structure sheaf \mathcal{O}_M . We denote by Θ the sheaf of germs of sections of the tangent bundle $T_J^{1,0}$ of $M = (X, J)$ and $\wedge^p \Theta$ is the sheaf of germs of p -th skew symmetric tensors of Θ . Our main theorem is the following :

THEOREM 0.1. *Let $M = (X, J)$ be a compact Kähler manifold. We assume that the direct sum of cohomology groups $\oplus_{i=0}^3 H^i(M, \wedge^{3-i} \Theta)$ vanishes. Then for every Kähler form ω and every non-zero holomorphic Poisson structure β , there exist deformations of bihermitian structures (J_t^+, J_t^-, h_t) which satisfies,*

$$J_t^+ = J_0^- = J, \quad \frac{d}{dt} J_t^-|_{t=0} = -2(\beta \cdot \omega + \bar{\beta} \cdot \omega), \quad (0.2)$$

where $\beta \cdot \omega$ is the $\bar{\partial}$ -closed forms of type $(0, 1)$ with coefficients in the tangent bundle $T_J^{1,0}$ which is given by the contraction between β and ω , and $\bar{\beta} \cdot \omega$ is the complex conjugate. The $\bar{\partial}$ -closed form $\beta \cdot \omega$ gives rise to the Kodaira-Spencer class $-2[\beta \cdot \omega] \in H^1(M, \Theta)$ of deformations $\{J_t^-\}$.

The condition (0.2) implies that $J_t^- \neq \pm J$ for small $t \neq 0$ as almost complex structures. However these J_t^- and J might be equivalent under diffeomorphisms. If the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ does not vanish, the family of deformations $\{J_t^-\}$ is not obtained by the action of a one-parameter family of diffeomorphisms on J . Thus the complex manifold (X, J_t^-) is different from (X, J) for small $t \neq 0$.

THEOREM 0.2. *Let $M = (X, J)$ be a compact Kähler manifold. We assume that the direct sum of cohomology groups $\oplus_{i=0}^3 H^i(M, \wedge^{3-i} \Theta)$ vanishes and in addition, the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ does not vanish for a Kähler form ω and a holomorphic Poisson structure β . Then there exist deformations of distinct bihermitian structures (J, J_t^-, h_t) , that is, (X, J_t^-) is not biholomorphic to $M = (X, J)$ for small $t \neq 0$.*

The infinitesimal deformations of generalized complex structures are given by the direct sum of cohomology group,

$$H^0(M, \wedge^2 \Theta) \oplus H^1(M, \Theta) \oplus H^2(M, \mathcal{O}_X),$$

where $H^1(M, \Theta)$ is the space of the Kodaira-Spencer classes which gives infinitesimal deformations of (usual) complex structures. The cohomology group $H^2(M, \mathcal{O}_X)$ corresponds to the exponential action of $\bar{\partial}$ -closed 2-form of type $(0, 2)$, which is often called

transformations by *b fields*. The space $H^0(M, \wedge^2 \Theta)$ corresponds to deformations of generalized complex structures $\{\mathcal{J}_{\beta t}\}$ by a Poisson structure β which are called *the Poisson deformations*. As in deformations of complex manifolds, there exists an obstruction to deformations of generalized complex structures in general. The obstruction space to deformations of generalized complex structures at \mathcal{J}_J is given by the direct sum of the ordinary cohomology groups,

$$\bigoplus_{i=0}^3 H^i(M, \wedge^{3-i} \Theta) := H^0(M, \wedge^3 \Theta) \oplus H^1(M, \wedge^2 \Theta) \oplus H^2(M, \Theta) \oplus H^3(M, \mathcal{O}_M)$$

which is the obstruction space in the theorem 0.1. If the space of the obstruction vanishes, we can apply the method in [10] and [11] to construct a family of generalized Kähler structures which corresponds to the one of bihermitian structures in the theorem 0.1. More precisely, the complex structure J gives a generalized complex structure \mathcal{J}_J and the Kähler structure ω also provides the d -closed non-degenerate, pure spinor $\psi = e^{\sqrt{-1}\omega}$ which induces the generalized complex structure \mathcal{J}_ψ . The pair (\mathcal{J}_J, ψ) gives rise to a generalized Kähler structure $(\mathcal{J}_J, \mathcal{J}_\psi)$. It is the essential feature that the generalized geometry inherits the symmetry of the Clifford group of the direct sum of the tangent bundle T and the cotangent bundle T^* on a manifold X . The space of almost generalized Kähler structures forms an orbit by the diagonal action of the Clifford group. Thus we construct deformations of almost generalized Kähler structures with one pure spinor starting with (\mathcal{J}_J, ψ) by the action of the Clifford group,

$$\mathcal{J}_t = \text{Ad}_{e^{Z(t)}} \mathcal{J}_J, \quad \psi_t = e^{Z(t)} \psi,$$

where $e^{Z(t)}$ is a family of the Clifford group and $\text{Ad}_{e^{Z(t)}}$ denotes the adjoint action of the Clifford group on \mathcal{J}_J (see [10]). The pair (\mathcal{J}_t, ψ_t) induces the almost generalized Kähler structure $(\mathcal{J}_t, \mathcal{J}_{\psi_t})$ and then the corresponding bihermitian structures (J_t^+, J_t^-, h_t) are given by the action of $\Gamma_t^\pm \in \text{GL}(TX)$ by $J_t^\pm = (\Gamma_t^\pm)^{-1} \circ J \circ \Gamma_t^\pm$, where Γ_t^\pm is explicitly described in terms of $Z(t)$ (see 3.4 in section 3). Thus our problem is reduced to construct $Z(t)$ which satisfies the following three conditions:

$$\mathcal{J}_t := \text{Ad}_{e^{Z(t)}} \mathcal{J}_J \text{ are integrable generalized complex structures} \quad (0.3)$$

$$d\psi_t := de^{Z(t)}\psi = 0 \quad (0.4)$$

$$(\Gamma_t^+)^{-1} \circ J \circ \Gamma_t^+ = J \quad (0.5)$$

Then $Z(t)$ yields deformations $(\mathcal{J}_t, \mathcal{J}_{\psi_t})$ of generalized Kähler structures which gives rise to bihermitian structures in the theorem 0.1 (see section 4 for more detail).

Note that if ψ_t is closed, the induced structure \mathcal{J}_{ψ_t} is integrable.

Hitchin [14] constructed deformations of bihermitian structure of the type (J, J_t^-, h_t) by the Hamiltonian diffeomorphisms on del Pezzo surfaces and Gualtieri [13] extended the approach to higher dimensional Poisson manifolds. The bihermitian structures which they constructed give the equivalent two complex structures under diffeomorphisms. Our constructions enable us to obtain distinct bihermitian structures.

In section 1, we will give a short explanation of deformations of generalized complex structures. Deformations of generalized complex structures are often described in the language of complex Lie algebroid [20], [12]. It is necessary to translate it in the terms of the action of the (real) Clifford group for our construction of generalized Kähler structures. In section 2, we recall the stability theorem of generalized Kähler structure with one pure spinor which was shown in [10] and [11]. In section 3, we give a description of Γ_t^\pm which gives deformations of bihermitian structures corresponding to the ones of generalized Kähler structures. In section 4, we will construct deformations of bihermitian structures in the main theorem 0.1 as formal power series. In section 5, we will show the convergence of the power series constructed in section 4 and finish our proof of the main theorem. In section 6, we apply our method to complex surfaces. In the case of complex surfaces, we only need to show that the cohomology groups $H^1(M, K_M^{-1})$ and $H^2(M, \Theta)$ vanish to obtain deformations in the theorem 0.1, where K_M is the canonical line bundle.

In subsection 6.1, we show that every del Pezzo surface admits deformations of bihermitian structures as in theorem 0.1. Let S_n be a del Pezzo surface which is the blow-up of \mathbb{CP}^2 at n points. Then we prove that if $n \geq 5$, there exists a class $[\beta \cdot \omega] \in H^1(S_n, \Theta)$ which does not vanish for a Kähler form ω . As a result, we obtain distinct bihermitian structures on S_n ($n \geq 5$). In subsection 6.2, we will give several vanishing theorems of $H^1(M, K_M^{-1})$ and $H^2(M, \Theta)$ on a complex surface M . In subsection 6.3 we will show the non-vanishing theorem of the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ which gives rise to unobstructed deformations. Applying these vanishing theorems and the non-vanishing theorem, we obtain bihermitian structure (J^+, J^-) on $F_2 = (X, J)$ on which the complex manifold (X, J^+) is F_2 and (X, J^-) is $\mathbb{CP}^1 \times \mathbb{CP}^1$ in subsection 6.4. We also show that the Hirtzebruch surface F_3 admits bihermitian structures. *Degenerate del Pezzo surfaces* are the blow-up of \mathbb{CP}^2 at r points, $0 \leq r \leq 8$ which are in *almost general position* (see subsection 6.5 for more detail). It turns out that the obstruction spaces still vanish on degenerate del Pezzo surfaces. If the anti-canonical line bundle is not ample, then there is a (-2) -curve C with $K \cdot C = 0$ and it follows that the class $[\beta \cdot \omega]$ does not vanish. Hence we obtain bihermitian structures in theorem 0.1 on the degenerate del Pezzo surfaces which yield distinct two complex manifolds. We contract all (-2) -curves on a degenerate del Pezzo

The author received a note that Gualtieri also developed a modified approach to obtain bihermitian structures on F_2 recently.

to obtain a del Pezzo surface with rational double points, which is called the *Gorenstein log del Pezzo surface*, [5, 4, 1]. In appendix I, we give the power series construction of the Kuranishi family of generalized complex structures. In appendix II, we collect necessary formulae and give an explanation of the Schouten bracket and the Jacobi identity of the brackets.

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1 Deformations of generalized complex structures

Let \mathcal{J} be a generalized complex structure on a compact manifold X of real dimension $2n$. Then the generalized complex structure \mathcal{J} gives the decomposition $(T \oplus T^*)^{\mathbb{C}} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$, where $L_{\mathcal{J}}$ denotes the eigenspace with eigenvalue $\sqrt{-1}$ and $\overline{L}_{\mathcal{J}}$ is the complex conjugate of $L_{\mathcal{J}}$. For a section $\varepsilon \in \wedge^2 \overline{L}_{\mathcal{J}}$, the exponential e^{ε} is regarded as the section of complex Clifford group which also induces the section of $\mathrm{SO}(T \oplus T^*, \mathbb{C})$ by the adjoint action $\mathrm{Ad}_{e^{\varepsilon}}$. Small deformations of almost generalized complex structure \mathcal{J} are written by a section of $\wedge^2 \overline{L}_{\mathcal{J}}$,

$$\mathcal{J}_t = \mathrm{Ad}_{e^{\varepsilon(t)}} \mathcal{J},$$

where t is the parameter of deformations. The integrability condition of almost generalized complex structure \mathcal{J}_t is given by the Maurer-Cartan equation,

$$d_L \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S = 0, \quad (1.1)$$

where $d_L : \wedge^k \overline{L}_{\mathcal{J}} \rightarrow \wedge^{k+1} \overline{L}_{\mathcal{J}}$ denotes the exterior derivative of the Lie algebroid $L_{\mathcal{J}}$ and the bracket $[\cdot, \cdot]_S$ is the Schouten bracket of $\overline{L}_{\mathcal{J}}$. Hence the problem of deformations reduces to solving the Maurer-Cartan equation,

At first, we write a family of sections $\varepsilon(t)$ as a power series in t

$$\varepsilon(t) = \varepsilon_1 t + \varepsilon_2 \frac{t^2}{2!} + \cdots, \quad (1.2)$$

Note that our power series starts from ε_1 . Substituting the power series (1.2) into the Maurer-Cartan equation, we obtain the equation on t . We denote by $([\varepsilon(t), \varepsilon(t)]_S)_{[k]}$ the k th homogeneous term in t . Then the equation is reduced to infinitely many equations,

$$\frac{1}{k!} d_L \varepsilon_k + \frac{1}{2} ([\varepsilon(t), \varepsilon(t)]_S)_{[k]} = 0 \quad (1.3)$$

We have the differential complex $(\wedge^{\bullet} \overline{L}_{\mathcal{J}}, d_L)$ which is elliptic,

$$0 \rightarrow \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \wedge^2 \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \wedge^3 \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \cdots$$

For $k = 1$, the equation is $d_L \varepsilon_1 = 0$. It implies that ε_1 is a section of $\wedge^2 \overline{L}_{\mathcal{J}}$ which is d_L -closed. We take ε_1 as a Harmonic section which satisfies

$$\Delta_L \varepsilon_1 = (d_L d_L^* + d_L^* d_L) \varepsilon_1 = 0,$$

where d_L^* is the formal adjoint operator of d_L with respect to a Riemannian metric on M . There are actions of diffeomorphisms and d -exact b -fields on generalized complex structures which generate d_L -exact sections of $\wedge^2 L_{\mathcal{J}}$ infinitesimally. We identify deformations by both actions of diffeomorphisms and d -exact b -fields. It implies that the infinitesimal deformations (the first order deformations) are given by the cohomology group $H^2(\overline{L}_{\mathcal{J}})$ of the elliptic differential complex $(\wedge^\bullet \overline{L}_{\mathcal{J}}, d_L)$. The third cohomology group $H^3(\overline{L}_{\mathcal{J}})$ is regarded as the space of the obstructions to deformations. The deformation theory of generalized complex structures was already discussed in [12] by the implicit function theorem. We will give the different construction of deformations of generalized complex structures by using the power series, which is analogous to the one of original Kodaira-Spencer theory. Our method yields an estimate of the convergent series which is necessary for the construction of generalized Kähler and bihermitian structures.

THEOREM 1.1. *If the cohomology group $H^3(\overline{L}_{\mathcal{J}})$ vanishes, then we have a family of deformations of generalized complex structures which are parametrized by an open set of $H^2(\overline{L}_{\mathcal{J}})$.*

PROOF. We solve the equations (1.3) by the induction on the degree of t . We assume that there are sections $\varepsilon_1, \dots, \varepsilon_{k-1} \in \wedge^2 \overline{L}_{\mathcal{J}}$ which satisfy the equations

$$\frac{1}{i!} \varepsilon_i + \frac{1}{2} ([\varepsilon(t), \varepsilon(t)]_S)_{[i]} = 0, \quad (1.4)$$

for all $i < k$.

Then we shall show that there is a section ε_k which satisfies the equation (1.3). The Schouten bracket and the Lie algebroid derivative d_L satisfy the following relations for sections $\varepsilon_1, \varepsilon_2 \in \wedge^\bullet \overline{L}_{\mathcal{J}}$,

PROPOSITION 1.2.

$$[\varepsilon_1, \varepsilon_2]_S = (-1)^{|\varepsilon_1| |\varepsilon_2|} [\varepsilon_2, \varepsilon_1]$$

$$d_L [\varepsilon_1, \varepsilon_2]_S = [d_L \varepsilon_1, \varepsilon_2]_S + (-1)^{|\varepsilon_1|} [\varepsilon_1, d_L \varepsilon_2]_S$$

$$(-1)^{|\varepsilon_1| |\varepsilon_3|} [[\varepsilon_1, \varepsilon_2], \varepsilon_3] + (-1)^{|\varepsilon_2| |\varepsilon_1|} [[\varepsilon_2, \varepsilon_3], \varepsilon_1] + (-1)^{|\varepsilon_3| |\varepsilon_1|} [[\varepsilon_3, \varepsilon_1], \varepsilon_2] = 0$$

where we denote by $|\varepsilon_i|$ the degree of ε_i .

The relations in the proposition 1.2 are already known [19]. Note that in our case the degree of ε_i is even and we have the ordinary Jacobi identity. The k th order term of the Schouten bracket $[\varepsilon(t), \varepsilon(t)]_S$ is given by

$$([\varepsilon(t), \varepsilon(t)]_S)_{[k]} = \sum_{\substack{i+j=k \\ 0 < i, j < k}} \frac{1}{i!} \frac{1}{j!} [\varepsilon_i, \varepsilon_j]_S. \quad (1.5)$$

Then substituting (1.4) into (1.5) and applying the proposition 1.2, we have

$$d_L ([\varepsilon(t), \varepsilon(t)]_S)_{[k]} = \sum_{\substack{i+j=k \\ 0 < i, j < k}} \frac{1}{i!} \frac{1}{j!} d_L [\varepsilon_i, \varepsilon_j]_S \quad (1.6)$$

$$= 2 \sum_{\substack{i+j=k \\ 0 < i, j < k}} \frac{1}{i!} \frac{1}{j!} [d_L \varepsilon_i, \varepsilon_j]_S \quad (1.7)$$

$$= - \sum_{\substack{l+m+j=k \\ 0 < l, m, j < k}} \frac{1}{l!} \frac{1}{m!} \frac{1}{j!} [[\varepsilon_l, \varepsilon_m]_S, \varepsilon_j]_S = 0 \quad (1.8)$$

Hence $([\varepsilon(t), \varepsilon(t)]_S)_{[k]} \in \wedge^2 \overline{L}_{\mathcal{J}}$ is d_L -closed which is a representative of the cohomology class in $H^3(\overline{L}_{\mathcal{J}})$. Since we assume that $H^3(\overline{L}_{\mathcal{J}})$ vanishes, we have a solution ε_k of the equation (1.3). We use a Riemannian metric on X to construct ε_k by using the formal adjoint operator d_L^* and the Green operator G_L of the elliptic complex $(\wedge^\bullet \overline{L}_{\mathcal{J}}, d_L)$

$$\frac{1}{k!} \varepsilon_k = -\frac{1}{2} d_L^* G_L ([\varepsilon(t), \varepsilon(t)]_S)_{[k]} \quad (1.9)$$

In fact, it follows from the Hodge decomposition theorem that ε_k satisfies the equation (1.3). Thus we obtain the solution $\varepsilon(t)$ of the Maurer-Cartan equation as a formal power series. In order to show the power series $\varepsilon(t)$ is a convergent series which is further a smooth solution, we apply the standard method due to Kodaira-Spencer. Let $P(t) = \sum_k P_k t^k$ be a power series in t whose coefficients are sections of a vector bundle with a metric. We denote by $\|P_k\|_s$ the Sobolev norm of the section P_k which is given by the sum of the L^2 -norms of i th derivative of P_k for all $i \leq s$, where s is a positive integer with $s > 2n + 1$. We put $\|P(t)\|_s = \sum_k \|P_k\|_s t^k$. Given two power series $P(t), Q(t)$, if $\|P_k\| \leq \|Q_k\|$ for all k , then we denote it by

$$P(t) \ll Q(t).$$

For a positive integer k , if $\|P_i\| \leq \|Q_i\|$ for all $i \leq k$, we write it by

$$P(t) \ll_k Q(t).$$

We also use the following notation. If $P_i = Q_i$ for all $i \leq k$, we write it by

$$P(t) \equiv_k Q(t). \quad (1.10)$$

Let $M(t)$ be a convergent power series defined by

$$M(t) = \sum_{\nu=1}^{\infty} \frac{1}{16c} \frac{(ct)^\nu}{\nu^2} = \sum_{\nu=1}^{\infty} M_\nu t^\nu, \quad (1.11)$$

for a positive constant c , which is determined later suitably. The key point is the following inequality,

$$M(t)^2 \ll \frac{1}{c} M(t) \quad (1.12)$$

We put $\lambda = c^{-1}$. Then we also have

$$e^{M(t)} \ll \frac{1}{\lambda} e^\lambda M(t). \quad (1.13)$$

We assume that our power series $\varepsilon(t)$ satisfies the inequality for an integer $k > 1$,

$$\|\varepsilon(t)\|_s \ll_{k-1} M(t) \quad (1.14)$$

We apply the standard estimate of elliptic differential operators to obtain an estimate of the solution $\varepsilon_{[k]}$ in (1.9),

$$\frac{2}{k!} \|\varepsilon_k\|_s \leq C_1 \|([\varepsilon(t), \varepsilon(t)]_S)_{[k]}\|_{s-1} = \sum_{\substack{i+j=k \\ 0 < i, j < k}} \frac{1}{i!} \frac{1}{j!} C_1 \|[\varepsilon_i, \varepsilon_j]_S\|_{s-1} \quad (1.15)$$

$$\leq 2C_1 \sum_{\substack{i+j=k \\ 0 < i, j < k}} \frac{1}{i!} \frac{1}{j!} \|\varepsilon_i\|_s \|\varepsilon_j\|_s = 2C_1 \sum_{\substack{i+j=k \\ 0 < i, j < k}} M_i M_j \quad (1.16)$$

$$\leq 2C_1 \lambda M_k \quad (1.17)$$

Hence if we choose a constant λ with $C_1 \lambda < 1$, then it follows that $\frac{1}{k!} \|\varepsilon_k\|_s < M_k$. It implies that $\varepsilon(t)$ is a convergent series. From our construction, the series $\varepsilon(t)$ satisfies

$$\varepsilon(t) = \varepsilon_1 t - \frac{1}{2} d_L^* G_L [\varepsilon(t), \varepsilon(t)]_S \quad (1.18)$$

Since $H^3(\wedge^\bullet \bar{L}_{\mathcal{J}}) = \{0\}$, we have a differential equation

$$\triangle_L \varepsilon(t) + \frac{1}{2} d_L^* [\varepsilon(t), \varepsilon(t)]_S = 0,$$

which is elliptic for sufficiently small $\varepsilon(t)$. Thus it follows that $\varepsilon(t)$ is a smooth solution. \square

In the appendix, we further construct the Kuranishi family of deformations of generalized complex structures which gives the space of deformations even in the cases where the obstruction space $H^3(\wedge^\bullet \overline{L}_{\mathcal{J}})$ does not vanish.

We denote by CL the Clifford algebra bundle of $T \oplus T^*$ on a manifold X which admits filtrations of even degree and odd degree,

$$\begin{aligned} \text{CL}^1 &\subset \text{CL}^3 \subset \text{CL}^5 \subset \dots \\ \text{CL}^0 &\subset \text{CL}^2 \subset \text{CL}^4 \subset \dots, \end{aligned}$$

where $\text{CL}^0 = T \oplus T^*$ and CL^2 denotes the subbundle of CL which consists of elements of degree 2 or 0, (for simplicity, we call CL the Clifford algebra of $T \oplus T^*$.) Let \mathcal{J} be a generalized complex structure on X which gives the decomposition,

$$(T \oplus T^*)^{\mathbb{C}} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$$

We denote by $\wedge^p \overline{L}_{\mathcal{J}}$ the bundle of the p th skew symmetric tensor of $\overline{L}_{\mathcal{J}}$. Let $U_{\mathcal{J}}^{-n}$ be the line bundle of (X, \mathcal{J}) which consists of non-degenerate, complex pure spinors corresponding to \mathcal{J} . We call $U_{\mathcal{J}}^{-n}$ the canonical line bundle $K_{\mathcal{J}}$ of \mathcal{J} . There is the action of $T \oplus T^*$ on differential forms $\wedge^\bullet T^*$ by the interior product and the exterior product which induces the spin representation of the Clifford algebra CL on $\wedge^\bullet T^*$. By the action of $\wedge^p \overline{L}_{\mathcal{J}}$ on $K_{\mathcal{J}}$, we have the vector bundles,

$$U^{-n+p} := \wedge^p \overline{L}_{\mathcal{J}} \cdot K_{\mathcal{J}}$$

Then the differential forms $\wedge^\bullet T^*$ on X are decomposed into

$$\wedge^\bullet T^* = \bigoplus_{p=0}^{2n} U^{-n+p}$$

We denote by $\pi_{U^{-n+p}}$ the projection to the bundle U^{-n+p} . The set of almost generalized complex structures forms the orbit of the (real) Clifford group of the Clifford algebra CL of $T \oplus T^*$ which acts on \mathcal{J} by the adjoint action. The Lie algebra of the Clifford group is the subalgebra CL^2 . Small deformations of almost generalized complex structures $\{\mathcal{J}_t\}$ are given in terms of the adjoint action,

$$\mathcal{J}_t := \text{Ad}_{e^{a(t)}} \mathcal{J},$$

where $a(t) = a_1 t + \frac{1}{2!} a_2 t^2 + \dots$ is a CL^2 -valued power series in t .

In order to obtain deformations of generalized Kähler structures, we need to consider a section $a(t)$ of the bundle $\text{CL}^2(T \oplus T^*)$.

it is crucial that the set of almost generalized Kähler structures just forms an orbit of the action of the real Clifford group and deformations of almost generalized Kähler

structures are not given by the action of the complex Clifford group. The following lemma is necessary for the construction of generalized Kähler structures. which is already proved in [10].

LEMMA 1.3. *For small deformations of almost generalized complex structures given by $\mathcal{J}_t := \text{Ad}_{e^{\varepsilon(t)}} \mathcal{J}_0$ as before, there exists a unique family of sections $a(t)$ of real Clifford bundle CL^2 such that*

$$\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}_0,$$

and $a(t)$ is in the real part of $\wedge^2 \overline{L}_{\mathcal{J}} \oplus \wedge^2 L_{\mathcal{J}}$. Conversely, If we have a family of deformations of almost generalized complex structure $\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}_0$ which is given by the action of a section $a(t) \in CL^2$, then there exists a unique section $\varepsilon(t) \in \wedge^2 \overline{L}_{\mathcal{J}}$ such that $\mathcal{J}_t = \text{Ad}_{e^{\varepsilon(t)}} \mathcal{J}_0$.

We consider the operator $e^{-a(t)} \circ d \circ e^{a(t)}$ acting on $K_J = U^{-n}$. Then as discussed in [9], the operator $e^{-a(t)} \circ d \circ e^{a(t)}$ is a Clifford-Lie operator of order 3 whose image is in $U^{-n+1} \oplus U^{-n+3}$.

It is shown in [10] that the almost generalized complex structure $\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}$ is integrable if and only if the projection to the component U^{-n+3} vanishes, that is,

$$\pi_{U^{-n+3}} e^{-a(t)} \circ d \circ e^{a(t)} = 0$$

We denote by $(\pi_{U^{-n+3}} e^{-a(t)} \circ d \circ e^{a(t)})_{[k]}$ the k th term of $\pi_{U^{-n+3}} e^{-a(t)} \circ d \circ e^{a(t)}$. Let \mathcal{J}_J be the generalized complex structure on X defined by a ordinary complex structure J . We put $M = (X, J)$. Then as in [10] and [11], the obstruction space to deformations of \mathcal{J}_J is given by $\oplus_{p+q=3} H^p(M, \wedge^q \Theta)$. In this case, the canonical line bundle is the ordinary one K_J which consists of complex forms of type $(n, 0)$. Thus by the theorem 1.1 and the lemma 1.3, we have the following,

PROPOSITION 1.4. *Let $M = (X, J)$ be a compact Kähler manifold with a Kähler form ω . We assume that the cohomology groups $\oplus_{p+q=3} H^p(M, \wedge^q \Theta)$ vanish. If there is a set of sections a_1, \dots, a_{k-1} of CL^2 which satisfies*

$$(\pi_{U^{-n+3}} e^{-a(t)} de^{a(t)})_{[i]} = 0, \quad \text{for all } i < k, \quad (1.19)$$

and $\|a(t)\|_s \ll_{k-1} C_1 M(t)$, then there is a section a_k of CL^2 which satisfies the followings:

$$\pi_{U^{-n+3}} (e^{-a(t)} de^{a(t)})_{[k]} = 0$$

and $\|a(t)\|_s \ll_k C_1 M(t)$, where $a(t) = \sum_{i=1}^{\infty} \frac{1}{i!} a_i t^i$ and $M(t)$ is the convergent series in (1.12) and C_1 is a positive constant.

PROOF. We use the notation as in (1.10). The equation (1.19) is equivalent to say that there is a section $\hat{E}(t) \in T \oplus T^*$ such that ,

$$e^{-a(t)} de^{a(t)} \cdot \phi \underset{k-1}{\equiv} \hat{E}(t) \cdot \phi, \quad (1.20)$$

for all $\phi \in K_J$. By the left action of $e^{a(t)}$ on both sides of the equation (1.20), we have

$$de^{a(t)} \cdot \phi \underset{k-1}{\equiv} e^{a(t)} \hat{E}(t) \cdot \phi.$$

We put $E(t) = e^{a(t)} \hat{E}(t) e^{-a(t)}$. Then it follows that

$$de^{a(t)} \cdot \phi \underset{k-1}{\equiv} E(t) \cdot e^{a(t)} \phi. \quad (1.21)$$

From the lemma 1.3, we have $\varepsilon(t) \in \wedge^2 \overline{L}_{\mathcal{J}}$ such that

$$e^{\varepsilon(t)} \cdot \phi \underset{k-1}{\equiv} e^{a(t)} \cdot \phi. \quad (1.22)$$

Substituting (1.22) into (1.21), we obtain

$$de^{\varepsilon(t)} \cdot \phi \underset{k-1}{\equiv} E(t) \cdot e^{\varepsilon(t)} \phi. \quad (1.23)$$

By the right action of $e^{-\varepsilon(t)}$ on (1.23) again, we have

$$e^{-\varepsilon(t)} de^{\varepsilon(t)} \cdot \phi \underset{k-1}{\equiv} e^{-\varepsilon(t)} E(t) \cdot e^{\varepsilon(t)} \phi \underset{k-1}{\equiv} \tilde{E}(t) \cdot \phi, \quad (1.24)$$

where $\tilde{E}(t) = e^{-\varepsilon(t)} E(t) \cdot e^{\varepsilon(t)}$. Thus as in [10], the equation(1.24) is equivalent to the Maurer-Cartan equation,

$$d_L \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S \underset{k-1}{\equiv} 0.$$

Then as is shown in the theorem 1.1, there is a section ε_k such that

$$d_L \varepsilon(t) + \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S \underset{k}{\equiv} 0.$$

We define a_k by $a_k = \varepsilon_k + \bar{\varepsilon}_k$. Then it follows that

$$e^{-a(t)} de^{a(t)} \cdot \phi \underset{k}{\equiv} \hat{E}(t) \cdot \phi.$$

Hence we have $(\pi_{U-n+3} e^{-a(t)} de^{a(t)})_{[k]} = 0$ and $\|a(t)\|_s \underset{k}{\ll} C_1 M(t)$. □

2 Deformations of generalized Kähler structures

Let (X, J, ω) be a compact Kähler manifold and $(\mathcal{J}, \mathcal{J}_\psi)$ the generalized Kähler structure induced from (J, ω) by $\mathcal{J} = \mathcal{J}_J$ and $\psi = e^{\sqrt{-1}\omega}$. Since two generalized complex structures \mathcal{J} and \mathcal{J}_ψ are commutative, the generalized Kähler structure $(\mathcal{J}, \mathcal{J}_\psi)$ gives the simultaneous decomposition of $(T \oplus T^*)^\mathbb{C}$,

$$(T \oplus T^*)^\mathbb{C} = L_{\mathcal{J}}^+ \oplus L_{\mathcal{J}}^- \oplus \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^-,$$

where $L_{\mathcal{J}}^+ \oplus L_{\mathcal{J}}^-$ is the eigenspace with eigenvalue $\sqrt{-1}$ with respect to \mathcal{J} and $L_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^-$ is the eigenspace with eigenvalue $\sqrt{-1}$ with respect to \mathcal{J}_ψ and $\overline{L}_{\mathcal{J}}^\pm$ denotes the complex conjugate. In [10, 11], the author showed the stability theorem of generalized Kähler structures with one pure spinor, which implies that if there is a one dimensional analytic deformations of generalized complex structures $\{\mathcal{J}_t\}$ parametrized by t , then there exists a family of non-degenerate, d -closed pure spinor ψ_t such that the family of pairs (\mathcal{J}_t, ψ_t) becomes deformations of generalized Kähler structures starting from $(\mathcal{J}, \psi) = (\mathcal{J}_0, \psi_0)$. As in section 2, small deformations \mathcal{J}_t can be written by the adjoint action of $a(t)$ in CL^2 ,

$$\mathcal{J}_t := \text{Ad}_{e^{a(t)}} \mathcal{J}_0.$$

Then we can obtain a family of real sections $b(t)$ of the bundle $(L_{\mathcal{J}_0}^- \cdot \overline{L}_{\mathcal{J}_0}^+ \oplus \overline{L}_{\mathcal{J}_0}^- \cdot L_{\mathcal{J}_0}^+)$ such that $\psi_t = e^{a(t)} e^{b(t)} \psi_0$ is the family of non-degenerate, d -closed pure spinor ψ_t . The bundle $K^1 = U^{0, -n+2}$ is generated by the action of real sections of $(L_{\mathcal{J}_0}^- \cdot \overline{L}_{\mathcal{J}_0}^+ \oplus \overline{L}_{\mathcal{J}_0}^- \cdot L_{\mathcal{J}_0}^+)$ on ψ (see page 125 in [11] for more detail).

We define $Z(t)$ by

$$e^{Z(t)} = e^{a(t)} e^{b(t)}.$$

Since $\text{Ad}_{e^{b(t)}} \mathcal{J}_0 = \mathcal{J}_0$, we obtain $\mathcal{J}_t = \text{Ad}_{e^{a(t)}} \mathcal{J}_0 = \text{Ad}_{e^{a(t)}} \text{Ad}_{e^{b(t)}} \mathcal{J}_0 = \text{Ad}_{e^{Z(t)}} \mathcal{J}_0$. Then the family of deformations of generalized Kähler structures is given by the action of $e^{Z(t)}$,

$$(\mathcal{J}_t, \psi_t) = (\text{Ad}_{e^{Z(t)}} \mathcal{J}_0, e^{Z(t)} \cdot \psi).$$

By the similar method as in [11] together with the proposition 1.4, we obtain the following proposition,

PROPOSITION 2.1. *Let (X, J, ω) be a compact Kähler manifold. We assume that the cohomology groups $\oplus_{p+q=3} H^p(X, \wedge^q \Theta)$ vanish. If there is a set of sections a_1, \dots, a_{k-1} of CL^2 which satisfies*

$$\pi_{U^{-n+3}} (e^{-a(t)} de^{a(t)})_{[i]} = 0, \quad \text{for all } i < k,$$

and $\|a(t)\|_s \ll_{k-1} K_1 M(t)$ for a positive constant K_1 , then there is a set of real sections b_1, \dots, b_k of the bundle $(L_{\mathcal{J}_0}^- \cdot \overline{L}_{\mathcal{J}_0}^+ \oplus \overline{L}_{\mathcal{J}_0}^- \cdot L_{\mathcal{J}_0}^+)$ which satisfies the following equations:

$$\pi_{U-n+3} (e^{-Z(t)} de^{Z(t)})_{[k]} = 0 \quad (2.1)$$

$$(de^{Z(t)} \cdot \psi_0)_{[i]} = 0, \quad \text{for all } i \leq k \quad (2.2)$$

$$\|a(t)\|_s \ll_k K_1 \lambda M(t) \quad (2.3)$$

$$\|b(t)\|_s \ll_k K_2 M(t) \quad (2.4)$$

where a_k is the section constructed in the proposition 1.4 and $e^{Z(t)} = e^{a(t)} e^{b(t)}$ and $M(t)$ is the convergent series in (1.12) and a positive constant K_2 is determined by λ and K_1 . The constant λ in $M(t)$ will be suitably selected to show the convergence of the power series $Z(t)$ in section 6.

3 Deformations of bihermitian structures

We use the same notation as in pervious sections. There is a one to one correspondence between generalized Kähler structures and bihermitian structures with the condition (0.1). In this section we shall give an explicit description of Γ_t^\pm which gives rise to bihermitian structure (J_t^+, J_t^-) corresponding to deformations (\mathcal{J}_t, ψ_t) . The correspondence is defined at each point on a manifold, that is, the correspondence between tensor fields which allows us to obtain almost bihermitian structures from almost generalized Kähler structures. The non-degenerate, pure spinor ψ_t induces the generalized complex structure \mathcal{J}_{ψ_t} . Since $(\mathcal{J}_t, \mathcal{J}_{\psi_t})$ is a generalized Kähler structure and \mathcal{J}_t commutes with \mathcal{J}_{ψ_t} , we have the simultaneous decomposition of $(T \oplus T^*)^\mathbb{C}$ into four eigenspaces,

$$(T \oplus T^*)^\mathbb{C} = L_{\mathcal{J}_t}^+ \oplus L_{\mathcal{J}_t}^- \oplus \overline{L}_{\mathcal{J}_t}^+ \oplus \overline{L}_{\mathcal{J}_t}^-,$$

where each eigenspace is given by the intersection of eigenspaces of both \mathcal{J}_t and \mathcal{J}_{ψ_t} ,

$$\begin{aligned} L_{\mathcal{J}_t}^- &= L_{\mathcal{J}_t} \cap \overline{L}_{\psi_t}, & \overline{L}_{\mathcal{J}_t}^+ &= \overline{L}_{\mathcal{J}_t} \cap \overline{L}_{\psi_t} \\ L_{\mathcal{J}_t}^+ &= L_{\mathcal{J}_t} \cap L_{\psi_t}, & \overline{L}_{\mathcal{J}_t}^- &= \overline{L}_{\mathcal{J}_t} \cap L_{\psi_t}, \end{aligned}$$

where $L_{\mathcal{J}_t}$ is the eigenspace of \mathcal{J}_t with eigenvalue $\sqrt{-1}$ and L_{ψ_t} denotes the eigenspace of \mathcal{J}_{ψ_t} with eigenvalue $\sqrt{-1}$. Since $\mathcal{J}_t = \text{Ad}_{e^{Z(t)}} \mathcal{J}_0 = e^{Z(t)} \mathcal{J}_0 e^{-Z(t)}$ and $\mathcal{J}_{\psi_t} = \text{Ad}_{e^{Z(t)}} \mathcal{J}_\omega$, we have the isomorphism between eigenspaces,

$$\text{Ad}_{e^{Z(t)}} : \overline{L}_{\mathcal{J}_0}^\pm \rightarrow \overline{L}_{\mathcal{J}_t}^\pm.$$

Let π be the projection from $T \oplus T^*$ to the tangent bundle T . We restrict the map π to the eigenspace $\overline{L}_{\mathcal{J}_0}^\pm$ which yields the map $\pi_t^\pm : \overline{L}_{\mathcal{J}_t}^\pm \rightarrow T^\mathbb{C}$. Let $T_{J_t^\pm}^{1,0}$ be the complex tangent

space of type $(1, 0)$ with respect to J_t^\pm . Then it follows that $T_{J_t^\pm}^{1,0}$ is given by the image of π_t^\pm ,

$$T_{J_t^\pm}^{1,0} = \pi_t^\pm(\overline{L_{\mathcal{J}_t}^\pm})$$

Since deformations of generalized Kähler structures are given by the action of $e^{Z(t)}$, the ones of bihermitian structures J_t^\pm should be described by the action of $\Gamma_t^\pm \in \text{GL}(T)$ which is obtained from $Z(t)$. We shall describe Γ_t^\pm in terms of $a(t)$ and $b(t)$. A local basis of $\overline{L_{\mathcal{J}_0}^\pm}$ is given by

$$\{\text{Ad}_{e^{\pm\sqrt{-1}\omega}} V_i = V_i \pm \sqrt{-1}[\omega, V_i]\}_{i=1}^n,$$

for a local basis $\{V_i\}_{i=1}^n$ of $T_J^{1,0}$, where we regard ω as an element of the Clifford algebra and then the bracket $[\omega, V_i]$ coincides with the interior product $i_{V_i}\omega$. It follows that the inverse map $(\pi_0^\pm)^{-1} : T_J^{1,0} \rightarrow \overline{L_{\mathcal{J}_0}^\pm}$ is given by the adjoint action of $e^{\pm\sqrt{-1}\omega}$,

$$\text{Ad}_{e^{\pm\sqrt{-1}\omega}} = (\pi_0^\pm)^{-1}. \quad (3.1)$$

We define a map $(\Gamma_t^\pm)^{1,0} : T_J^{1,0} \rightarrow T_{J_t^\pm}^{1,0}$ by the composition,

$$(\Gamma_t^\pm)^{1,0} = \pi_t^\pm \circ \text{Ad}_{e^{Z(t)}} \circ (\pi_0^\pm)^{-1} \quad (3.2)$$

$$= \pi \circ \text{Ad}_{e^{Z(t)}} \circ \text{Ad}_{e^{\pm\sqrt{-1}\omega}} \quad (3.3)$$

$$\begin{array}{ccc} \overline{L_{\mathcal{J}_0}^\pm} & \xrightarrow{\text{Ad}_{e^{Z(t)}}} & \overline{L_{\mathcal{J}_t}^\pm} \\ \pi_0^\pm \downarrow & & \downarrow \pi_t^\pm \\ T_J^{1,0} & \xrightarrow{(\Gamma_t^\pm)^{1,0}} & T_{J_t^\pm}^{1,0} \end{array}$$

Together with the complex conjugate $(\Gamma_t^\pm)^{0,1} : T_J^{0,1} \rightarrow T_{J_t^\pm}^{0,1}$, we obtain the map Γ_t^\pm which satisfies $J_t^\pm = (\Gamma_t^\pm)^{-1} \circ J \circ \Gamma_t^\pm$.

Let J^* be the complex structure on the cotangent space T^* which is given by $\langle J^*\eta, v \rangle = \langle \eta, Jv \rangle$, where $\eta \in T^*$ and $v \in T$ and \langle, \rangle denote the coupling between T and T^* . We define a map $\hat{J}^\pm : T \oplus T^* \rightarrow T \oplus T^*$ by $\hat{J}^\pm(v, \eta) = v \mp J^*\eta$ for $v \in T$ and $\eta \in T^*$. Then Γ_t^\pm is written as

$$\Gamma_t^\pm = \pi \circ \text{Ad}_{e^{Z(t)}} \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}. \quad (3.4)$$

The k th term of Γ_t^\pm is denoted by $(\Gamma_t^\pm)_{[k]}$ as before. Note that $(\Gamma_t^\pm)_{[0]} = \text{id}_T$. We also put $\Gamma_t^\pm(a(t), b(t)) = \Gamma_t^\pm$.

LEMMA 3.1. *The k th term $(\Gamma_t^\pm)_{[k]}$ is given by*

$$(\Gamma_t^\pm)_{[k]} = \frac{1}{k!} \pi \circ (\text{ad}_{a_k} + \text{ad}_{b_k}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega} + \widetilde{\Gamma}_k^\pm(a_{<k}, b_{<k})$$

where the second term $\widetilde{\Gamma}_k^\pm(a_{<k}, b_{<k})$ depends only on a_1, \dots, a_{k-1} and b_1, \dots, b_{k-1} .

PROOF. Substituting the identity $\text{Ad}_{e^{Z(t)}} = \text{id} + \text{ad}_{Z(t)} + \frac{1}{2!}(\text{ad}_{Z(t)})^2 + \dots$, we have

$$\Gamma_t^\pm = \pi \circ \text{Ad}_{e^{Z(t)}} \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega} \quad (3.5)$$

$$= \pi \circ \left(\sum_{i=0}^{\infty} \frac{1}{i!} \text{ad}_{Z(t)}^i \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega} \right) \quad (3.6)$$

$$(3.7)$$

Then k -th term is given by

$$(\Gamma_t^\pm)_{[k]} = \pi \circ \left(\text{ad}_{Z(t)} \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega} \right)_{[k]} + \sum_{i=2}^k \pi \circ \left(\frac{1}{i!} (\text{ad}_{Z(t)}^i \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}) \right)_{[k]} \quad (3.8)$$

$$= \frac{1}{k!} \pi \circ (\text{ad}_{a_k} + \text{ad}_{b_k}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega} + \widetilde{\Gamma}_k^\pm(a_{<k}, b_{<k}), \quad (3.9)$$

where $\widetilde{\Gamma}_k^\pm(a_{<k}, b_{<k})$ denotes the non-linear term depending a_1, \dots, a_{k-1} and b_1, \dots, b_{k-1} . \square

LEMMA 3.2. *Let b be a section of the bundle $(L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$. Then we have*

$$[\pi(\text{ad}_b \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}), J] = 0 \in \text{End}(T).$$

PROOF. Applying (3.1), for $v \in T_J^{1,0}$, we obtain

$$\hat{J}^\pm \circ \text{Ad}_{e^\omega} v = \text{Ad}_{e^{\pm\sqrt{-1}\omega}} v = (\pi_0^\pm)^{-1} v \in \overline{L}_{\mathcal{J}}^\pm.$$

Since $\text{ad}_b(\overline{L}_{\mathcal{J}}^\pm) = [b, \overline{L}_{\mathcal{J}}^\pm] \subset \overline{L}_{\mathcal{J}}^\mp$ and $\pi(\overline{L}_{\mathcal{J}}^\mp) = T_J^{1,0}$, thus we have $\pi(\text{ad}_b \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}) v \in T_J^{1,0}$. It follows that $[\pi(\text{ad}_b \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}), J] = 0$. \square

The tensor space $T \otimes T^*$ defines a subbundle of CL^2 . We denote it by $T \cdot T^*$. An element $\gamma \in T \cdot T^*$ gives the endmorphism ad_γ by $\text{ad}_\gamma E = [\gamma, E]$ for $E \in T \oplus T^*$, which preserves the cotangent bundle T^* .

LEMMA 3.3. *Let γ be an element of $T \cdot T^*$. Then we have*

$$\pi \circ (\text{ad}_\gamma \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}) = \text{ad}_\gamma \in \text{End}(T).$$

PROOF. For a tangent vector $v \in T$, we have $\text{Ad}_{e^\omega} v = v + [\omega, v] = v + \text{ad}_\omega v$. Since the map ad_γ preserves the cotangent T^* , we have $\text{ad}_\gamma \circ \hat{J}^\pm \circ \text{ad}_\omega(v) \in T^*$ for all tangent $v \in T$. Thus it follows that $\pi(\text{ad}_\gamma \circ \hat{J}^\pm \circ \text{ad}_\omega) = 0$, since π is the projection to the tangent T . Thus we obtain the result. \square

LEMMA 3.4. We assume that there is a set of sections $a_1, \dots, a_k \in \text{CL}^2$ and real sections $b_1, \dots, b_k \in (L_{\mathcal{J}_0}^- \cdot \overline{L}_{\mathcal{J}_0}^+ \oplus \overline{L}_{\mathcal{J}_0}^- \cdot L_{\mathcal{J}_0}^+)$ which satisfies the following equations,

$$\begin{aligned} \pi_{U^{-n+3}} (e^{-Z(t)} d e^{Z(t)})_{[i]} &= 0, & 0 \leq \forall i \leq k \\ (d e^{Z(t)} \cdot \psi_0)_{[i]} &= 0, & 0 \leq \forall i \leq k \\ [(\Gamma_t^\pm)_{[i]}, J] &= 0, & 0 \leq \forall i < k \end{aligned}$$

Then the k -th term $(\Gamma_t^\pm)_{[k]}$ satisfies

$$\pi_{U^{-n+3}} [d, (\Gamma_t^\pm)_{[k]}] = 0,$$

where $[d, (\Gamma_t^\pm)_{[k]}]$ is an operator from $U^{-n} = K_{\mathcal{J}}$ to $U^{-n+1} \oplus U^{-n+3}$ and $\pi_{U^{-n+3}}$ denotes the projection to the component U^{-n+3} .

PROOF. Since we assume that the space of the obstructions to deformations of generalized complex structures vanishes, we obtain a family of section $\check{a}(t)$ with $\check{a}_i = a_i$ for $i = 1, \dots, k$ such that $\check{a}(t)$ gives deformations of generalized complex structures, that is,

$$\pi_{U^{-n+3}} e^{-\check{a}(t)} d e^{\check{a}(t)} = 0.$$

The the stability theorem of generalized Kähler structures in [10] provides deformations of generalized Kähler structures with one pure spinor, $(\text{Ad}_{e^{\check{Z}(t)}} \mathcal{J}_0, e^{\check{Z}(t)} \psi_0)$, where $e^{\check{Z}(t)} = e^{\check{a}(t)} e^{\check{b}(t)}$, where $\check{b}(t)$ is a family of real sections with $\check{b}_i = b_i$, for $i = 1, \dots, k$. From the correspondence between generalized Kähler structures and bihermitian structures, we have the family of bihermitian structures (J_t^+, J_t^-) which is given by the action of $\check{\Gamma}_t^\pm := \Gamma_t^\pm(\check{a}(t), \check{b}(t))$ of $\text{GL}(T)$. Since J_t^\pm is integrable, we have

$$\pi_{U^{-n+3}} ((\check{\Gamma}_t^\pm)^{-1} d \check{\Gamma}_t^\pm) = 0. \quad (3.10)$$

Let Ω be a d -closed form of type $(n, 0)$ which is a local basis of $K_J = K_{\mathcal{J}}$. Then as in the argument of proof of the proposition 1.4, we have

$$d\Gamma_t^\pm \Omega \equiv_k \Gamma_t^\pm E(t) \Omega.$$

Since $d\Omega = 0$, the degree of $E(t)$ is greater than or equal to 1. The condition $[(\Gamma_t^\pm)_{[i]}, J] = 0$ ($0 \leq i < k$) implies that $(\Gamma_t^\pm)_{[i]} E(t) \Omega \in U_{\mathcal{J}}^{-n+1}$. Thus we have

$$d(\Gamma_t^\pm)_{[k]} \Omega = \sum_{\substack{i+j=k \\ 0 \leq i, j < k}} (\Gamma_t^\pm)_{[i]} E(t)_{[j]} \Omega \in U_{\mathcal{J}}^{-n+1}$$

Hence we have $\pi_{U^{-n+3}} [d, (\Gamma_t^+)_{[k]}] = 0$. □

4 Construction of deformations of bihermitian structures with $J_t^+ = J$

This section and next section are devoted to prove our main theorem 0.1. We use the same notation as before. As we see in the lemma 3.1, $(\Gamma_t^\pm)_{[k]}$ depends on a_1, \dots, a_k and b_1, \dots, b_k . We write $\Gamma_t^\pm(a_{<k}, a_k, b_{<k}, b_k)$ for $(\Gamma_t^\pm)_{[k]}$.

Let β be a holomorphic 2-vector field on a compact Kähler manifold (X, J, ω) , that is, β is a section of $\wedge^2 \Theta = \wedge^2 T_J^{1,0}$. For β , we shall construct a section $a(t) \in \text{CL}^2$ and a real section $b(t) \in (L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ such that the action of the family of the Clifford group

$$e^{Z(t)} = e^{a(t)} e^{b(t)}$$

on $(\mathcal{J}, \psi) = (\mathcal{J}_0, \psi_0)$ gives rise to a family of generalized Kähler structures $(\text{Ad}_{e^{Z(t)}} \mathcal{J}_J, e^{Z(t)} \psi)$ which satisfies the following three conditions:

$$\mathcal{J}_t := \text{Ad}_{e^{Z(t)}} \mathcal{J}_J \text{ are integrable generalized complex structures} \quad (4.1)$$

$$d\psi_t := de^{Z(t)} \psi = 0 \quad (4.2)$$

$$J_t^+ = J, \quad (4.3)$$

where (J_t^+, J_t^-) denote the corresponding bihermitian structures. It follows from (4.1), (4.2) that (\mathcal{J}_t, ψ_t) are generalized Kähler structures with one pure spinor which give rise to deformations of bihermitian structures preserving J_t^+ from (4.3). Let K_J be the canonical line bundle on (X, J) which consists on holomorphic n -forms. The action CL^1 on K_J provides a bundle $\text{CL}^1 \cdot K_J$. Then as before, the condition (4.1) is equivalent to the followings,

$$e^{-Z(t)} de^{Z(t)} \cdot K_J \subset \text{CL}^1 \cdot K_J, \quad (4.4)$$

This implies that the $e^{-Z(t)} de^{Z(t)} \cdot \Omega$ is written as $E \cdot \Omega$ for any form Ω of type $(n, 0)$, where $E \in \text{CL}^1 = T \oplus T^*$. As we see in the previous section, the condition (4.3) is equivalent to $[\Gamma_t^+(a(t), b(t)), J] = 0$. We denote by $(e^{-Z(t)} de^{Z(t)})_{[i]}$ the i -th term of $(e^{-Z(t)} de^{Z(t)})$ on t and also write i -th terms of $d\psi_t$ and Γ_t^+ by $(d\psi_t)_{[i]}$ and $(\Gamma_t^+)_{[i]}$ respectively. Then the three equations (4.1), (4.2) and (4.3) are reduced to the following system of equations,

$$(e^{-Z(t)} de^{Z(t)})_{[i]} \cdot K_J \subset \text{CL}^1 \cdot K_J, \quad 0 \leq \text{for all } i \leq k \quad (4.5)$$

$$(d\psi_t)_{[i]} := (de^{Z(t)} \psi)_{[i]} = 0, \quad 0 \leq \text{for all } i \leq k \quad (4.6)$$

$$[(\Gamma_t^+(a(t), b(t)))_{[i]}, J] = 0, \quad 0 \leq \text{for all } i \leq k \quad (4.7)$$

We shall construct a solution of the system of the equations by the induction on degree

k of t . In the first case $k = 1$, three equations are given by

$$\begin{aligned} (e^{-Z(t)} d e^{Z(t)})_{[1]} \cdot K_J &= [d, a_1] \cdot K_J \subset \text{CL}^1 \cdot K_J \\ (d\psi_t)_{[1]} &= d(a_1 + b_1) \cdot \psi_0 = 0 \\ [(\Gamma_t^+(a_1, b_1))_{[1]}, J] &= 0 \end{aligned}$$

At first we put $\hat{a}_1 = \beta + \bar{\beta}$, where $\bar{\beta}$ denotes the complex conjugate of β . Since β is holomorphic, it follows that $[d, \hat{a}_1] \cdot K_J \subset \text{CL} \cdot K_J$. Then from the proposition 2.1, we have a real section $\hat{b}_1 \in (L_{\mathcal{J}}^- \cdot \bar{L}_{\mathcal{J}}^+ \oplus \bar{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ with $d(\hat{a}_1 + \hat{b}_1) \cdot \psi_0 = 0$. Then $\Gamma_t^\pm(\hat{a}_1, \hat{b}_1)$ is given by

$$\Gamma_t^\pm(\hat{a}_1, \hat{b}_1) = \pi \circ (\text{ad}_{\hat{a}_1} + \text{ad}_{\hat{b}_1}) \circ \hat{J}^\pm \circ \text{Ad}_{e^\omega}.$$

Then we define $\gamma_1 \in T \cdot T^*$ by

$$\text{ad}_{\gamma_1} = - \left(\Gamma_t^+(\hat{a}_1, \hat{b}_1) \right)_{[1]} \in \text{End}(T). \quad (4.8)$$

It follows from the lemma 3.4 that γ_1 satisfies $[d, \gamma_1] \cdot K_J \subset \text{CL}^1 \cdot K_J$, where we identify $\text{End}(T)$ with $T \cdot T^*$. We define a_1 by

$$a_1 = \hat{a}_1 + \gamma_1. \quad (4.9)$$

Then we have

$$(e^{-Z(t)} d e^{Z(t)})_{[1]} \cdot K_J = [d, a_1] \cdot K_J \quad (4.10)$$

$$= ([d, \hat{a}_1] + [d, \gamma_1]) \cdot K_J \subset \text{CL}^1 \cdot K_J \quad (4.11)$$

From the proposition 2.1, we also have a real section $b_1 \in (L_{\mathcal{J}}^- \cdot \bar{L}_{\mathcal{J}}^+ \oplus \bar{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ with $d(a_1 + b_1) \cdot \psi_0 = 0$. Applying the lemma 3.1 and substituting a_1 and b_1 into $(\Gamma_t^+)_{[1]}$, we have

$$(\Gamma_t^+(a_1, b_1))_{[1]} = \pi \circ (\text{ad}_{a_1} + \text{ad}_{b_1}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.12)$$

$$= \pi \circ (\text{ad}_{\hat{a}_1} + \text{ad}_{\gamma_1} + \text{ad}_{\hat{b}_1} + \text{ad}_{b_1 - \hat{b}_1}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.13)$$

Applying the lemma 3.3 to γ_1 and using (4.8), we obtain

$$(\Gamma_t^+(a_1, b_1))_{[1]} = \text{ad}_{\gamma_1} + \pi \circ (\text{ad}_{\hat{a}_1} + \text{ad}_{\hat{b}_1}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.14)$$

$$+ \pi \circ \text{ad}_{b_1 - \hat{b}_1} \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.15)$$

$$= \text{ad}_{\gamma_1} + \Gamma_t^+(\hat{a}_1, \hat{b}_1) + \pi \circ \text{ad}_{b_1 - \hat{b}_1} \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.16)$$

$$= \pi \circ \text{ad}_{b_1 - \hat{b}_1} \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.17)$$

Since $b_1 - \hat{b}_1 \in (L_{\mathcal{J}}^- \cdot \bar{L}_{\mathcal{J}}^+ \oplus \bar{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$, it follows from the lemma 3.2 that

$$[(\Gamma_t^+(a_1, b_1))_{[1]}, J] = 0.$$

Hence a_1 and b_1 as above satisfies the three equations for $k = 1$.

We assume that there is a set of real sections $a_1, \dots, a_{k-1} \in \text{CL}^2$ and $b_1, \dots, b_{k-1} \in (L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ which satisfies the system of equations:

$$(e^{-Z(t)} d e^{Z(t)})_{[i]} \cdot K_J \subset \text{CL}^1 \cdot K_J, \quad 0 \leq \text{for all } i \leq k-1 \quad (4.18)$$

$$(d\psi_t)_{[i]} := (de^{Z(t)}\psi)_{[i]} = 0, \quad 0 \leq \text{for all } i \leq k-1 \quad (4.19)$$

$$[(\Gamma_t^+(a(t), b(t)))_{[i]}, J] = 0, \quad 0 \leq \text{for all } i \leq k-1 \quad (4.20)$$

The k -th term $(e^{-Z(t)} d e^{Z(t)})_{[k]} \cdot K_J$ is decomposed into the linear term $\frac{1}{k!}[d, a_k] \cdot K_J$ and the nonlinear term $\text{Ob}_k^J(a_{<k}, b_{<k}) \cdot K_J$ which is called the term of the obstruction,

$$(e^{-\hat{Z}(t)} d e^{\hat{Z}(t)})_{[k]} \cdot K_J = \frac{1}{k!}[d, a_k] \cdot K_J + \text{Ob}_k^J(a_{<k}, b_{<k}) \cdot K_J.$$

We also have the decomposition of the k -th term $(de^{\hat{Z}(t)}\psi)_{[k]}$,

$$(de^{\hat{Z}(t)}\psi)_{[k]} = \frac{1}{k!}d(a_k + b_k)\psi + \text{Ob}_k^{\psi_0}(a_{<k}, b_{<k})$$

From the proposition 2.1, we have the sections \hat{a}_k and $\hat{b}_k \in (L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ which satisfies

$$\left(\frac{1}{k!}[d, \hat{a}_k] + \text{Ob}_k^J(a_{<k}, b_{<k}) \right) \cdot K_J \subset \text{CL}^1 \cdot K_J, \quad (4.21)$$

$$\frac{1}{k!}d(\hat{a}_k + \hat{b}_k)\psi_0 + \text{Ob}_k^{\psi_0}(a_{<k}, b_{<k}) = 0 \quad (4.22)$$

Then from the lemma 3.1, $\Gamma_t^+(a_{<k}, \hat{a}_k, b_{<k}, b_k)$ is given by

$$\Gamma_t^+(a_{<k}, \hat{a}_k, b_{<k}, b_k) = \pi \circ (\text{ad}_{\hat{a}_k} + \text{ad}_{\hat{b}_k}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} + \tilde{\Gamma}_k^+(a_{<k}, b_{<k}).$$

Then we define $\gamma_k \in T \cdot T^*$ by using \hat{a}_k and \hat{b}_k

$$\text{ad}_{\gamma_k} = - \left(\Gamma_t^+(a_{<k}, \hat{a}_k, b_{<k}, \hat{b}_k) \right)_{[k]}. \quad (4.23)$$

It follows from the lemma 3.4 that we have $[d, \gamma_k] \cdot K_J \subset \text{CL}^1 \cdot K_J$. We define a_k by

$$a_k = \hat{a}_k + \gamma_k. \quad (4.24)$$

Then we have

$$(e^{-Z(t)} d e^{Z(t)})_{[k]} \cdot K_J = \left(\frac{1}{k!}[d, a_k] + \text{Ob}_k^J(a_{<k}, b_{<k}) \right) \cdot K_J \quad (4.25)$$

$$= \left(\frac{1}{k!}[d, \gamma_k] + \frac{1}{k!}[d, \hat{a}_k] + \text{Ob}_k^J(a_{<k}, b_{<k}) \right) \cdot K_J \subset \text{CL}^1 \cdot K_J, \quad (4.26)$$

where $e^{Z(t)} = e^{a(t)} e^{b(t)}$. We apply the proposition 2.1 again to obtain a real section $b_k \in (L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ which satisfies $(de^{Z(t)} \cdot \psi)_{[k]} = 0$. Then from the lemma 3.1 and (4.24), $(\Gamma_t^+(a(t), b(t)))_{[k]}$ is given by

$$k! (\Gamma_t^+(a(t), b(t)))_{[k]} = \pi (\text{ad}_{a_k} + \text{ad}_{b_k}) \circ J^* \circ \text{Ad}_{e^\omega} + k! \widetilde{(\Gamma_k^\pm)}(a_{<k}, b_{<k}) \quad (4.27)$$

$$= \pi (\text{ad}_{\hat{a}_k} + \text{ad}_{\gamma_k} + \text{ad}_{b_k}) \circ J^* \circ \text{Ad}_{e^\omega} + k! \widetilde{(\Gamma_k^\pm)}(a_{<k}, b_{<k}) \quad (4.28)$$

$$(4.29)$$

Applying lemma 3.3 to γ_k and using (4.23), we obtain

$$k! (\Gamma_t^+(a(t), b(t)))_{[k]} = \text{ad}_{\gamma_k} + \pi (\text{ad}_{\hat{a}_k} + \text{ad}_{b_k}) \circ J^* \circ \text{Ad}_{e^\omega} \quad (4.30)$$

$$+ k! \widetilde{(\Gamma_k^\pm)}(a_{<k}, b_{<k}) + \pi (\text{ad}_{b_k - \hat{b}_k}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.31)$$

$$= \text{ad}_{\gamma_k} + \Gamma^+(a_{<k}, \hat{a}_k, b_{<k}, \hat{b}_k)_{[k]} + \pi (\text{ad}_{b_k - \hat{b}_k}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.32)$$

$$= \pi (\text{ad}_{b_k - \hat{b}_k}) \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \quad (4.33)$$

Since $b_k - \hat{b}_k \in (L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$, it follows from the lemma 3.2 that

$$[(\Gamma_t^+(a(t), b(t)))_{[k]}, J] = 0.$$

Hence the set of sections a_k, b_k together with $a_{<k}, b_{<k}$ satisfies three equations (4.5), (4.6) and (4.7). Thus from our assumption of the induction, we successively solve the equations to obtain a set of sections $a(t)$ and $b(t)$ which satisfies three equations (4.5), (4.6) and (4.7) for all k . The solution $(a(t), b(t))$ is given in the form of a formal power series in t . Next section we shall show that both $a(t)$ and $b(t)$ are convergent series which are smooth.

Our construction is well explained by the following figure,

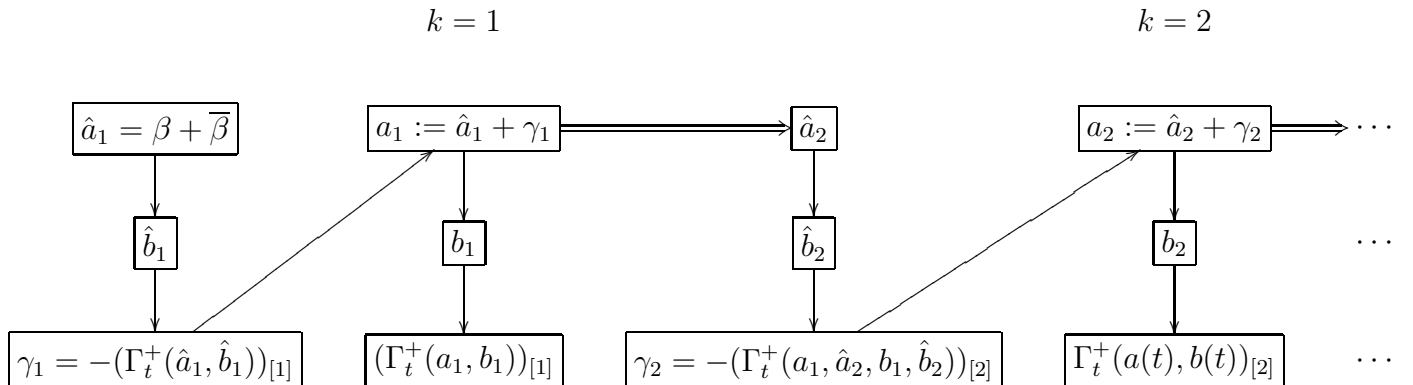


Figure 1

5 The convergence

As in the proposition 2.1, if there is a set of sections a_1, \dots, a_{k-1} of CL^2 which satisfies

$$\pi_{U^{-n+3}} (e^{-a(t)} de^{a(t)})_{[i]} = 0, \quad \text{for all } i < k,$$

and $\|a(t)\|_s \ll_{k-1} K_1 M(t)$, then there is a set of real sections $b_1, \dots, b_k \in (L_{\mathcal{J}}^- \cdot \overline{L}_{\mathcal{J}}^+ \oplus \overline{L}_{\mathcal{J}}^- \cdot L_{\mathcal{J}}^+)$ which satisfy the following equations:

$$\pi_{U^{-n+3}} (e^{-Z(t)} de^{Z(t)})_{[k]} = 0 \quad (5.1)$$

$$(de^{Z(t)} \cdot \psi_0)_{[i]} = 0, \quad \text{for all } i \leq k \quad (5.2)$$

$$\|\hat{a}_k\|_s < K_1 \lambda M_k \quad (5.3)$$

$$\|\hat{b}_k\|_s < K_2 M_k \quad (5.4)$$

where \hat{a}_k is the section in the proposition 1.4 and $M(t)$ is the convergent series in (1.12) with a constant λ and K_1 is a positive constant and a positive constant K_2 is determined by λ, K_1 . We also have an estimate of $e^{Z(t)} = e^{a(t)} e^{b(t)}$ in [10],

$$\|Z(t)\| \ll_k M(t).$$

Then γ_k in (4.23) satisfies

$$\|\gamma_k\|_s < \|\Gamma_k^+(a_{<k}, \hat{a}_k, b_{<k}, \hat{b}_k)\|_s \quad (5.5)$$

$$< 2\|\hat{a}_k\|_s + 2\|\hat{b}_k\|_s + \|\widetilde{\Gamma}_k^+(a_{<k}, b_{<k})\|_s \quad (5.6)$$

Recall that $\Gamma_t^+ = \pi \left(\text{Ad}_{e^{Z(t)}} \circ \hat{J}^+ \circ \text{Ad}_{e^\omega} \right)$. Then we have an estimate of the non-linear term $\|\widetilde{\Gamma}_k^+(a_{<k}, b_{<k})\|_s$

$$\|\widetilde{\Gamma}_k^+(a_{<k}, b_{<k})\|_s < C \|(e^{Z(t)} - Z(t) - 1)_{[k]}\|_s,$$

where C denotes a constant. It follows from (1.13) that $\|(e^{Z(t)} - Z(t) - 1)_{[k]}\|_s < C(\lambda) M_k$, where $C(\lambda)$ satisfies $\lim_{\lambda \rightarrow 0} C(\lambda) = 0$. Thus we have

$$\|\gamma_k\|_s < 2\|\hat{a}_k\|_s + 2\|\hat{b}_k\|_s + C(\lambda) M_k < 2\lambda K_1 M_k + K_2 M_k + C(\lambda) M_k$$

We take λ and K_2 sufficiently small such that $3\lambda K_1 M_k + K_2 M_k + C(\lambda) M_k < K_1 M_k$. Then we obtain

$$\|a_k\|_s < \|\hat{a}_k\|_s + \|\gamma_k\|_s < K_1 M_k.$$

Thus our solution $a(t)$ satisfies that $\|a(t)\|_s \ll_k K_1 M(t)$ for all k . It implies that $a(t)$ is a convergent series. Applying the proposition 2.1 again, we have $\|b(t)\|_s <_k K_2 M(t)$. Hence $b(t)$ is also a convergent series. Thus it follows that $Z(t)$ is a convergent series.

PROOF. OF THEOREM 0.1 AND THEOREM 0.2. The sections $a(t)$ and $b(t)$ which constructed in section 5 give deformations of bihermitian structures (J_t^+, J_t^-) . We shall show that the family of deformations satisfies the condition in the theorem 0.1. We already have $[\Gamma_t^+, J] = 0$ which implies that $J_t^+ = J$. From the lemma 3.1 and the lemma 3.2, the 1st term of J_t^- is given by

$$\begin{aligned} [(\Gamma_t^-)_{[1]}, J] &= [\pi \circ (\text{ad}_{\hat{a}_1} + \text{ad}_{\gamma_1} + \text{ad}_{\hat{b}_1}) \circ \hat{J}^- \circ \text{Ad}_{e^\omega}, J] \\ &= [(\text{ad}_{\gamma_1} + \pi \circ \text{ad}_{\hat{a}_1} \circ \hat{J}^- \circ \text{Ad}_{e^\omega}), J] \end{aligned}$$

Since $\hat{a}_1 = \beta + \bar{\beta}$, we have $\pi \circ \text{ad}_{\hat{a}_1}|_T = 0$. We also have $\text{ad}_{\gamma_1} = -\Gamma^+(\hat{a}_1, \hat{b}_1)$ and $[\text{ad}_{\gamma_1}, J] = [(\pi \circ \text{ad}_{\gamma_1} \circ J^* \circ \text{ad}_\omega), J]$. Thus we obtain

$$[(\Gamma_t^-)_{[1]}, J] = 2[(\pi \circ \text{ad}_{\hat{a}_1} \circ J^* \circ \text{ad}_\omega), J]$$

Then we have for a vector v ,

$$\begin{aligned} 2(\pi \circ \text{ad}_{\hat{a}_1} \circ J^* \circ \text{ad}_\omega)v &= -2[\beta + \bar{\beta}, [\omega, Jv]] \\ &= -2[\beta + \bar{\beta}, \omega], Jv = -2(\beta \cdot \omega + \bar{\beta} \cdot \omega)Jv. \end{aligned}$$

Thus it follows that $\frac{d}{dt}J_t^-|_{t=0} = [(\Gamma_t^-)_{[1]}, J] = -2(\beta \cdot \omega + \bar{\beta} \cdot \omega)$ and the Kodaira-Spencer class of deformations $\{J_t^-\}$ is given by the class $-2[\beta \cdot \omega] \in H^1(M, \Theta)$. If the Kodaira-Spencer class does not vanish, then the deformations $\{J_t^-\}$ is not trivial. Thus (X, J_t^-) is not biholomorphic to (X, J) for small $t \neq 0$. \square

6 Applications

6.1 Bihermitian structures on del Pezzo surfaces

A del Pezzo surface is by definition a smooth algebraic surface with ample anti-canonical line bundle. A classification of del Pezzo surfaces are well known, they are $\mathbb{CP}^1 \times \mathbb{CP}^1$ or \mathbb{CP}^2 or a surface S_n which is the blow-up of \mathbb{CP}^2 at n points P_1, \dots, P_n , ($0 < n \leq 8$). The set of the points $\Sigma := \{P_1, \dots, P_n\}$ must be in *general position* to yield a del Pezzo surface. The following theorem is due to Demazure, [4] (see page 27), which shows the meaning of *general position*,

THEOREM 6.1. *The following conditions are equivalent:*

- (1) *The anti-canonical line bundle of S_n is ample*
- (2) *No three of Σ lie on a line, no six of Σ lie on a conic and no eight of Σ lie on a cubic with a double point $P_i \in \Sigma$*
- (3) *There is no curve C on S_n with $-K_{S_n} \cdot C \leq 0$.*
- (4) *There is no curve C with $C \cdot C = -2$ and $K_{S_n} \cdot C = 0$.*

REMARK 6.2. If three points lie on a line l , then the strict transform \hat{l} of l in S_3 is a (-2) -curve with $K_{S_3} \cdot \hat{l} = 0$. If six points belong to a conic curve C , then the strict transform \hat{C} of C is again a (-2) -curve with $K_{S_6} \cdot \hat{C} = 0$. If eight points $P_1 \cdots, P_8$ lie on a cubic curve with a double point P_1 , then the strict transform \hat{C} of C satisfies $\hat{C} \sim \pi^{-1}C - 2E_1 - E_2 - \cdots - E_8$, where E_i is the exceptional curve $\pi^{-1}(P_i)$. Then we also have $\hat{C}^2 = -2$ and $K_{S_8} \cdot \hat{C} = 0$.

Let D be a smooth anti-canonical divisor of S_n which is given by the zero locus of a section $\beta \in H^0(S_n, K_{S_n}^{-1})$. Since the anti-canonical bundle $K_{S_n}^{-1}$ is regarded as the bundle of 2-vectors $\wedge^2 \Theta$ and $[\beta, \beta]_S = 0 \in \wedge^3 \Theta$ on S_n , every section β is a holomorphic Poisson structure. On S_n , we have the followings,

$$\dim H^1(S_n, \Theta) = \begin{cases} 2n - 8 & (n = 5, 6, 7, 8) \\ 0 & (n < 5) \end{cases}$$

$$\dim H^0(S_n, K^{-1}) = 10 - n$$

and

$$H^{1,1}(S_n) = 1 + n.$$

Further we have $H^2(S_n, \Theta) = \{0\}$, $H^1(S_n, \wedge^2 \Theta) \cong H^1(S_n, -K_{S_n}) = \{0\}$. Hence the obstruction vanishes and we have deformations of generalized complex structures parametrized by $H^0(S_n, K_{S_n}^{-1}) \oplus H^1(S_n, \Theta)$.

In particular, if $n \geq 5$, we have deformations of ordinary complex structures on S_n

PROPOSITION 6.3. *Let D be a smooth anti-canonical divisor given by the zero locus of β as above. Then there is a Kähler form ω with the class $[\beta \cdot \omega] \neq 0 \in H^1(S_n, \Theta)$.*

We also have $H^2(\mathbb{CP}^1 \times \mathbb{CP}^1, \Theta) = 0$ and $H^1(\mathbb{CP}^1 \times \mathbb{CP}^1, -K) = 0$.

Thus we can apply our construction to every del Pezzo surface. From the main theorem 0.1 together with the proposition 6.3, we have

PROPOSITION 6.4. *Every del Pezzo surface admits deformations of bihermitian structures (J, J_t^-, h_t) with $J_0^- = J$ which satisfies*

$$\frac{d}{dt} J_t^-|_{t=0} = -2(\beta \cdot \omega + \bar{\beta} \cdot \omega), \quad (6.1)$$

for every Kähler form ω and every holomorphic Poisson structure β . Further, a del Pezzo surface S_n ($n \geq 5$) admits distinct bihermitian structures (J, J_t^-, h_t) , that is, the complex manifold (X, J_t^-) is not biholomorphic to (X, J) for small $t \neq 0$.

Note that for small $t \neq 0$, $J_t^- \neq \pm J$. We will give a proof of the proposition 6.3 in the rest of this subsection.

Let N_D is the normal bundle to D in S_n and $i^*T_{S_n}$ the pull back of the tangent bundle T_{S_n} of S_n by the inclusion $i : D \rightarrow S_n$. Then we have the short exact sequence,

$$0 \rightarrow T_D \rightarrow i^*T_{S_n} \rightarrow N_D \rightarrow 0$$

and we have the long exact sequence

$$0 \rightarrow H^0(D, T_D) \rightarrow H^0(D, i^*T_{S_n}) \rightarrow H^0(D, N_D) \xrightarrow{\delta} H^1(D, T_D) \rightarrow \dots$$

Since the line bundle N_D is positive, $H^1(D, N_D) = \{0\}$ and $\dim H^0(D, N_D)$ is equal to the intersection number $D \cdot D = 9 - n$ by the Riemann-Roch theorem. Since D is an elliptic curve, $\dim H^1(D, T_D) = \dim H^0(D, T_D) = 1$. Hence it follows that

$$9 - n \leq \dim H^0(D, i^*T_{S_n}) \leq 10 - n. \quad (6.2)$$

Let \mathcal{I}_D be the ideal sheaf of D and \mathcal{O}_D the structure sheaf of D . Then we have the short exact sequence

$$0 \rightarrow \mathcal{I}_D \rightarrow \mathcal{O}_{S_n} \rightarrow i_*\mathcal{O}_D \rightarrow 0$$

By the tensor product, we also have

$$0 \rightarrow \mathcal{I}_D \otimes T_{S_n} \rightarrow T_{S_n} \rightarrow i_*\mathcal{O}_D \otimes T_{S_n} \rightarrow 0 \quad (6.3)$$

Then from the projection formula we have

$$H^p(S_n, i_*\mathcal{O}_D \otimes T_{S_n}) \cong H^p(S_n, i_*(\mathcal{O}_D \otimes i^*T_{S_n})) \cong H^p(D, i^*T_{S_n}),$$

for $p = 0, 1, 2$. From (6.3), we have the long exact sequence,

$$H^0(S_n, T_{S_n}) \rightarrow H^0(D, i^*T_{S_n}) \rightarrow H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) \xrightarrow{j} H^1(S_n, T_{S_n}) \rightarrow \dots \quad (6.4)$$

Hence we obtain

LEMMA 6.5. *The map $j : H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) \rightarrow H^1(S_n, T_{S_n})$ is not the zero map .*

PROOF. We have the exact sequence,

$$\dots \rightarrow H^0(D, i^*T_{S_n}) \rightarrow H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) \xrightarrow{j} H^1(S_n, T_{S_n}) \quad (6.5)$$

From the Serre duality with $\mathcal{I}_D = K_{S_n}$, we have $H^0(S_n, \mathcal{I}_D \otimes T_{S_n}) \cong H^2(S_n, \Omega_{S_n}^1) = \{0\}$ and $H^2(S_n, \mathcal{I}_D \otimes T_{S_n}) = H^0(S_n, \Omega^1) = 0$. From the Riemann-Roch theorem, $\dim H^1(S_n, \mathcal{I}_D \otimes T_{S_n}) = n + 1$. Then it follows from (6.2) that

$$\dim H^0(D, i^*T_{S_n}) < \dim H^1(S_n, \mathcal{I}_D \otimes T_{S_n})$$

Note $10 - n < n + 1$ for all $n \geq 5$. Hence the map j is non zero. \square

REMARK 6.6. Since $n \geq 5$, we have $H^0(S_n, T_{S_n}) = \{0\}$. Applying the Serre duality with $K_{S_n} = \mathcal{I}_D$, we have $H^2(S_n, T_{S_n}) \cong H^0(S_n, \mathcal{I}_D \otimes \Omega^1) = 0$. From the Riemann-Roch, we obtain $\dim H^1(S_n, T_{S_n}) = 2n - 8$.

Let β be a non-zero holomorphic Poisson structure S_n with the smooth divisor D as the zero locus. Then β is regarded as a section of $\mathcal{I}_D \otimes \wedge^2 \Theta$. Thus the section $\beta \in H^0(S_n, \mathcal{I}_D \otimes \wedge^2 \Theta)$ gives an identification,

$$\Omega^1 \cong \mathcal{I}_D \otimes T_{S_n}.$$

Then the identification induces the isomorphism

$$\hat{\beta} : H^1(S_n, \Omega^1) \cong H^1(S_n, \mathcal{I}_D \otimes T_{S_n}).$$

Let j be the map in the lemma 6.5. Then we have the composite map $j \circ \hat{\beta} : H^1(S_n, \Omega^1) \rightarrow H^1(S_n, \Theta)$ which is given by the class $[\beta \cdot \omega] \in H^1(S_n, T_{S_n})$ for $[\omega] \in H^1(S_n, \Omega^1)$.

PROPOSITION 6.7. *The composite map $j \circ \hat{\beta} : H^1(S_n, \Omega^1) \rightarrow H^1(S_n, T_{S_n})$ is not the zero map.*

PROOF. Since the map $\hat{\beta}$ is an isomorphism, $\hat{\beta}(\omega)$ is not zero. It follows from lemma 6.5 that the map j is non-zero. Hence the composite map $j \circ \hat{\beta}$ is non-zero also. \square

PROOF. of lemma 6.3 The set of Kähler class is an open cone in $H^{1,1}(S_n, \mathbb{R}) \cong H^2(S_n, \mathbb{R})$. We have the non-zero map $j \circ \hat{\beta} : H^2(S_n, \mathbb{C}) \cong H^1(S_n, \Omega^1) \rightarrow H^1(S_n, \Theta)$ for each $\beta \in H^0(S_n, K^{-1})$ with $\{\beta = 0\} = D$. It follows that the kernel $j \circ \hat{\beta}$ is a closed subspace and the intersection $\ker(j \circ \hat{\beta}) \cap H^2(S_n, \mathbb{R})$ is closed in $H^2(S_n, \mathbb{R})$ whose dimension is strictly less than $\dim H^2(S_n, \mathbb{R})$. Thus the complement in the Kähler cone

$$\{[\omega] : \text{Kähler class} \mid j \circ \hat{\beta}([\omega]) \neq 0\}$$

is not empty. Thus there is a Kähler form ω such that the class $[\beta \cdot \omega] \in H^1(S_n, \Theta)$ does not vanish for $n \geq 5$. \square

We also remark that our proof of the lemma 6.3 still works for degenerate del Pezzo surfaces.

6.2 Vanishing theorems on surfaces

Let M be a compact complex surface with canonical line bundle K_M . We shall give some vanishing theorems of the cohomology groups $H^1(M, -K_M)$ and $H^2(M, \Theta)$ on a compact smooth complex surface M , which are the obstruction spaces to deformations of generalized complex structures starting from the ordinary one (X, \mathcal{J}_J) . The following is practical to show the vanishing of $H^1(M, -K_M)$.

PROPOSITION 6.8. *Let M be a compact complex surface with $H^1(M, \mathcal{O}_M) = 0$. If $-K_M = m[D]$ for a irreducible, smooth curve D with positive self-intersection number $D \cdot D > 0$ and a positive integer m , then $H^1(M, K_M^n) = 0$ for all integer n .*

The proposition is often used in the complex geometry. For completeness, we give a proof.

PROOF. Let I_D be the ideal sheaf of the curve D . Then we have the short exact sequence, $0 \rightarrow I_D \rightarrow \mathcal{O}_M \rightarrow j_*\mathcal{O}_D \rightarrow 0$, where $j : D \rightarrow X$. Then we have the exact sequence,

$$H^0(M, \mathcal{O}_M) \rightarrow H^0(M, j_*\mathcal{O}_D) \xrightarrow{\delta} H^1(M, I_D) \rightarrow H^1(M, \mathcal{O}_M)$$

It follows that the coboundary map δ is a 0-map. Thus from $H^1(M, \mathcal{O}_M) = 0$, we have $H^1(M, I_D) = H^1(M, -[D]) = 0$. We use the induction on k . We assume that $H^1(M, I_D^k) = H^1(M, -k[D]) = 0$ for a positive integer k . The short exact sequence $0 \rightarrow I_D^{k+1} \rightarrow I_D^k \rightarrow j_*\mathcal{O}_D \otimes I_D^k \rightarrow 0$ induces the exact sequence,

$$H^0(M, j_*\mathcal{O}_D \otimes I_D^k) \rightarrow H^1(M, I_D^{k+1}) \rightarrow H^1(M, I_D^k).$$

By the projection formula, we have $H^0(M, j_*\mathcal{O}_D \otimes I_D^k) = H^0(D, -k[D]|_D)$. Since $D \cdot D > 0$, it follows that the line bundle $-k[D]|_D$ is negative and then $H^0(D, -k[D]|_D) = H^0(M, I_D^k) = 0$. It implies that $H^1(M, I_D^{k+1}) = H^1(M, -(k+1)[D]) = 0$. Thus by the induction, we have $H^1(M, -nD) = 0$ for all positive integer n . Applying the Serre duality, we have $H^1(M, -nD) \cong H^1(M, (n-m)D) = 0$. Thus $H^1(M, nD) = 0$ for all integer n . Then the result follows since $H^1(M, K^n) = H^1(M, -(nm)D) = 0$. \square

The author also refer to the standard vanishing theorem. If $D = \sum_i a_i D_i$ is a \mathbb{Q} -divisor on M , where D_i is a prime divisor and $a_i \in \mathbb{Q}$. Let $\lceil a_i \rceil$ be the round-up of a_i and $\lfloor a_i \rfloor$ the round-down of a_i . Then the fractional part $\{a_i\}$ is $a_i - \lfloor a_i \rfloor$. Then the round-up and the round-down of D is defined by

$$\lceil D \rceil = \sum_i \lceil a_i \rceil D_i, \quad \lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$$

and $\{D\} = \sum_i \{a_i\} D_i$ is the fractional part of D . A divisor D is **nef** if one has $D \cdot C \geq 0$ for any curve C . A divisor D is **nef** and **big** if in addition, one has $D^2 > 0$. We shall use the following vanishing theorem. The two dimensional case is due to Miyaoka and the higher dimensional cases are due to Kawamata and Viehweg

THEOREM 6.9. *Let M be a smooth projective surface and D a \mathbb{Q} -divisor on M such that*

- (1) *$\text{supp}\{D\}$ is a divisor with normal crossings,*
- (2) *D is nef and big.*

Then $H^i(M, K_M + \lceil D \rceil) = 0$ for all $i > 0$.

If $-K_M = mD$ is nef and big divisor where D is smooth for $m > 0$. Then applying the theorem, we have

$$H^i(M, -K_M) \cong H^i(M, K_M - 2K_M) = 0,$$

for all $i > 0$.

Next we consider the vanishing of the cohomology group $H^2(M, \Theta)$. Applying the Serre duality theorem, we have

$$H^2(M, \Theta) \cong H^0(M, \Omega^1 \otimes K_M)$$

If $-K_M$ is an effective divisor $[D]$, then K_M is given by the ideal sheaf I_D of D . The short exact sequence: $0 \rightarrow \Omega^1 \otimes I_D \rightarrow \Omega^1 \rightarrow \Omega^1 \otimes \mathcal{O}_D \rightarrow 0$ gives us the injective map,

$$0 \rightarrow H^0(M, \Omega^1 \otimes K_M) \rightarrow H^0(M, \Omega^1).$$

Hence we have

PROPOSITION 6.10. *if M is a smooth surface with effective anti-canonical divisor satisfying $H^0(M, \Omega^1) = 0$, then we have the vanishing $H^2(M, \Theta) = 0$.*

6.3 Non-vanishing theorem

PROPOSITION 6.11. *Let M be a Kähler surface with a Kähler form ω and a non-zero Poisson structure $\beta \in H^0(M, \wedge^2 \Theta)$. Let D be the divisor defined by the section β . If there is a curve C of M with $C \cap \text{supp } D = \emptyset$, then the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ does not vanish.*

PROOF. Since β is not zero on the complement $M \setminus D$, there is a holomorphic symplectic form $\hat{\beta}$ on the complement. The symplectic form $\hat{\beta}$ gives the isomorphism $\Theta \cong \Omega^1$ on $M \setminus D$ which induces the isomorphism between cohomology groups $H^1(M \setminus D, \Theta) \cong H^1(M \setminus D, \Omega^1)$. Then the restricted class $[\beta \cdot \omega]|_{M \setminus D}$ corresponds to the Kähler class $[\omega]|_{M \setminus D} \in H^1(M \setminus D, \Omega^1) \cong H^{1,1}(M \setminus D)$ under the isomorphism. Since there is the curve C on the complement $M \setminus D$ and ω is a Kähler form, the class $[\omega|_C] \in H^{1,1}(C)$ does not vanish. Then it follows that the class $[\omega]|_{M \setminus D} \in H^1(M \setminus D, \Omega^1)$ does not vanish. It implies that $[\beta \cdot \omega]|_{M \setminus D}$ does not vanish also. Thus we have that the class $[\beta \cdot \omega] \in H^1(M, \Theta)$ does not vanish. \square

6.4 Deformations of bihermitian structures on the Hirtzebruch surfaces F_2 and F_3

Let F_2 be the projective space bundle of $T^*\mathbb{CP}^1 \oplus \mathcal{O}_{\mathbb{CP}^1}$,

$$F_2 = \mathbb{P}(T^*\mathbb{CP}^1 \oplus \mathcal{O}_{\mathbb{CP}^1}).$$

We denote by E^+ and E^- the sections of F_2 with positive and negative self-intersection numbers respectively. An anti-canonical divisor of F_2 is given by $2E^+$, while the section E^- with $E^- \cdot E^- = -2$ is the curve which satisfies $E^+ \cap E^- = \emptyset$. Thus we have the non-vanishing class $[\beta \cdot \omega] \in H^1(F_2, \Theta)$, where β is a section of $-K$ with the divisor $2E^+$. (Note that the canonical holomorphic symplectic form $\hat{\beta}$ on the cotangent bundle $T^*\mathbb{CP}^1$ which induces the holomorphic Poisson structure β . The structure β can be extended to F_2 which gives the anti-canonical divisor $2[E^+]$.)

PROPOSITION 6.12. *The class $[\beta \cdot \omega] \in H^1(F_2, \Theta)$ does not vanish for every Kähler form ω on F_2 .*

PROOF. The result follows from the proposition 6.11. \square

On the surface F_2 , the anti-canonical line bundle of F_2 is $2E^+$ and $H^1(F_2, \mathcal{O}_{F_2}) = 0$. Hence from the proposition 6.8, we have the vanishing $H^i(F_2, -K_X) = \{0\}$ for all $i > 0$. Since the surface F_2 is simply connected, it follows from the proposition 6.10 that $H^2(F_2, \Theta) = 0$. Hence the obstruction vanishes and we can apply our main theorem. It is known that every non-trivial small deformation of F_2 is $\mathbb{CP}^1 \times \mathbb{CP}^1$. Thus we have

PROPOSITION 6.13. *Let (X, J) be the Hirtzebruch surface F_2 as above. Then there is a family of deformations of bihermitian structures (J_t^+, J_t^-, h_t) with $J_t^+ = J_0^+ = J$ such that (X, J_t^-) is $\mathbb{CP}^1 \times \mathbb{CP}^1$ for small $t \neq 0$.*

Let F_e be the projective space bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ over \mathbb{CP}^1 with $e > 0$. There is a section b with $b^2 = -e$, which is unique if $e > 0$. Let f be a fibre of F_e . Then $-K$ is given by $2b + (e + 2)f$, which is an effective divisor. Thus from the proposition 6.10, we have $H^2(F_e, \Theta) = \{0\}$. $P_{-1}(F_e) = \dim H^0(F_e, K^{-1})$ is listed in the table 7.1.1 of [24],

$$P^{-1}(F_e) = \begin{cases} 9 & e = 0, 1 \\ 9 & e = 2 \\ e + 6 & e \geq 3 \end{cases}$$

Since K is given by the ideal sheaf I_D for the effective divisor $D = 2b + (e + 2)f$, It follows from the Serre duality that $H^2(F_e, K^{-1}) = H^0(F_e, I_D^2) = \{0\}$. Thus applying the Riemann-Roch theorem, we obtain

$$\dim H^1(F_e, K^{-1}) = e - 3,$$

for $e \geq 3$. In the case $e = 3$, we have $H^1(F_3, K^{-1}) = H^2(F_3, \Theta) = \{0\}$. Thus from the theorem 0.1, we have

PROPOSITION 6.14. *The Hirtzebruch surface F_3 admits deformations of bihermitian structures (J, J_t^-, h_t) with $J_t^- \neq \pm J$ for small $t \neq 0$.*

We can generalize our discussion of F_2 to the projective space bundle of $T^*M \oplus \mathcal{O}_M$ over a compact Kähler manifold M . Then we also have the Poisson structure β and as in the proposition 6.11, it is shown that the class $[\beta \cdot \omega]$ does not vanish. Thus we have the deformations of bihermitian structures from the stability theorem [11]. For the ones as in the theorem 0.1, we need to show the vanishing of the obstruction. Note that the obstruction space does not vanish in general.

If M is a Riemannian surface Σ_g of genus $g \geq 1$, then the projective space bundle is called a ruled surface of degree g . It is known that small deformations of any ruled surface of degree $g \geq 1$ remain to be ruled surfaces of the same degree. Applying the stability theorem, we have

PROPOSITION 6.15. *Let (X, J) be a ruled surface $\mathbb{P}(T^*\Sigma_g \oplus \mathcal{O}_{\Sigma_g})$ with degree $g \geq 1$. Then there is a family of non-trivial bihermitian structures (J_t^+, J_t^-, h_t) such that $J \neq \pm J_t^\pm$ and (X, J_t^\pm) is a ruled surface for small t .*

6.5 Bihermitian structures on degenerate del Pezzo surfaces

We shall consider the blow-up of $\mathbb{C}P^2$ at r points which are not in general position. We follow the construction as in [4], (see page 36). We have a finite set $\Sigma = \{x_1, \dots, x_r\}$ and $X(\Sigma)$ obtained by successive blowing up at Σ ,

$$X(\Sigma) \rightarrow X(\Sigma_{r-1}) \rightarrow \dots \rightarrow X(\Sigma_1) \rightarrow \mathbb{C}P^2,$$

At first $X(\Sigma_1)$ is the blow-up of $\mathbb{C}P^2$ at a point $x_1 \in \mathbb{C}P^2$ and we have $\Sigma_i = \{x_1, \dots, x_i\}$ and $X(\Sigma_{i+1})$ is the blow-up of $X(\Sigma_i)$ at $x_{i+1} \in X(\Sigma_i)$. Let E_i be the divisor given by the inverse image of $x_i \in X(\Sigma_{i-1})$. If Γ is an effective divisor on $\mathbb{C}P^2$, one notes that $\text{mult}(x_i, \Gamma)$ the multiplicity of x_i on the proper transform of Γ in $X(\Sigma_{i-1})$, and one says that Γ passes through x_i if $\text{mult}(x_i, \Gamma) > 0$. Define $\hat{E}_1, \dots, \hat{E}_r$ by recurrence as follows, On $X(\Sigma_1)$, one put $\hat{E}_1 = E_1$; on $X(\Sigma_2)$, \hat{E}_1 is a proper transform of the previous E_1 and one also put $\hat{E}_2 = E_2$; on $X(\Sigma_3)$, \hat{E}_1 and \hat{E}_2 are the proper transform of previous \hat{E}_1 and \hat{E}_2 respectively and $\hat{E}_3 = E_3$. Then $\hat{E}_1, \dots, \hat{E}_r$ are irreducible components of $E_1 + \dots + E_r$.

We assume that the following condition on Σ ,

(*) For each $i = 1, \dots, r$, a point $x_i \in X(\Sigma_{i-1})$ does not belong to a irreducible curve \hat{E}_j with self-intersection number -2 for $1 \leq j \leq i-1$.

If a point $x_i \in X(\Sigma_{i-1})$ belongs to a irreducible curve \hat{E}_j with self-intersection number -2 , then the proper transform of \hat{E}_j becomes a curve with self-intersection number -3 . If there is a rational curve with self-intersection number -3 or less, the anti-canonical divisor of $X(\Sigma)$ is not nef.

DEFINITION 6.16. A set of points Σ is in *almost general position* if Σ satisfies the following:

- (1) Σ satisfies the condition (*)
- (2) No line passes through 4 points of Σ
- (3) No conic passes through 7 points of Σ

We call $X(\Sigma)$ a *degenerate del Pezzo surface* if Σ is in almost general position. Note that if Σ is in general position, Σ is in almost general position. In [4], the following theorem was shown,

THEOREM 6.17. [4] *The following conditions are equivalent:*

- (1) Σ is in almost general position
- (2) The anti-canonical class of $X(\Sigma)$ contains a smooth and irreducible curve D .
- (3) There is a smooth curve of \mathbb{CP}^2 passing all points of Σ .
- (4) $H^1(X(\Sigma), K_{X(\Sigma)}^n) = \{0\}$ for all integer n
- (5) $-K_{X(\Sigma)} \cdot C \geq 0$ for all effective curve C on $X(\Sigma)$ and in addition, if $-K_{X(\Sigma)} \cdot C = 0$, then $C \cdot C = -2$.

Then from (2) there is a smooth anti-canonical divisor on a degenerate del Pezzo surface and we have $H^1(X(\Sigma), \mathcal{O}_X) = 0$. Hence from the proposition 6.8, we have the vanishing $H^i(X(\Sigma), -K_X) = 0$, for all $i > 0$. A degenerate del Pezzo surface $X(\Sigma)$ satisfies $H^0(X(\Sigma), \Omega^1) = 0$. Then it follows from the proposition 6.10 that $H^2(X(\Sigma), \Theta) = 0$.

Let $X(\Sigma)$ be a degenerate del Pezzo surface which is not a del Pezzo surface, that is, the anti-canonical class of $X(\Sigma)$ is not ample. Then from (5), there is a (-2) -curve C with $K_{X(\Sigma)} \cdot C = 0$. Then it follows that C is a \mathbb{CP}^1 . Thus we contract (-2) -curves on a degenerate del Pezzo to obtain a complex surface with rational double points, which is called the Gorenstein log del Pezzo surface. Let β be a section of $-K_X$ with the smooth divisor D as the zero set. We denote by J the complex structure of the del Pezzo surface $X(\Sigma)$. From the theorem 0.1, we have

THEOREM 6.18. *A degenerate del Pezzo surface admits deformations of distinct bi-hermitian structures (J, J_t^-, h_t) with $J_0^- = J$ and $J_t^- \neq \pm J$ for small $t \neq 0$, that is, $\frac{d}{dt} J_t^-|_{t=0} = -2(\beta \cdot \omega + \bar{\beta} \cdot \omega)$, and the complex structure J_t^- is not equivalent to J of $X(\Sigma)$ under diffeomorphisms for small $t \neq 0$, where ω is a Kähler form.*

PROOF. If $X(\Sigma)$ is a del Pezzo surface, we already have the result. If $X(\Sigma)$ is not a del Pezzo but a degenerate del Pezzo, we still have $H^2(X(\Sigma), \Theta) = H^1(X(\Sigma), K^{-1}) = \{0\}$. Thus we have deformations of bihermitian structures as in the theorem 0.1. It is sufficient to show that the class $[\beta \cdot \omega]$ does not vanish. Since $K \cdot C = 0$, the line bundle $K|_C \rightarrow C \cong \mathbb{CP}^1$ is trivial. If there is a point $P \in D \cap C$, then $\beta(P) = 0$ and it follows that $\beta|_C \equiv 0$. Since D is smooth, we have $D = C$. However $D \cdot D = 9 - r$ and $D \cdot C = -K \cdot C = 0$. Thus $D \cap C = \emptyset$. Then applying the proposition 6.11, we obtain $[\beta \cdot \omega] \neq 0 \in H^1(X(\Sigma), \Theta)$. \square

7 Appendix I (The Kuranishi family of generalized complex structures)

We shall discuss an analog of the Kuranishi family of deformations of generalized complex structures. The deformation theory of generalized complex structures was already obtained in [12] by using the implicit function theorem. For the completeness of this paper, we will give the different construction of deformations of generalized complex structures by using the power series. Our method explicitly shows that the deformations family depends holomorphically on the parameter t and we can also have an estimate of the convergent series as in section 1.

Let (X, \mathcal{J}) be a compact generalized complex manifold and $\overline{L} := \overline{L}_{\mathcal{J}}$ the Lie algebroid bundle as before which gives the decomposition, $(T \oplus T^*)^{\mathbb{C}} = L \oplus \overline{L}$. Note that the obstruction space $H^3(\wedge^{\bullet} \overline{L}_{\mathcal{J}})$ does not necessary vanish. Even in the case we obtain the family of deformations which is parametrized by an analytic set. We fix a metric on X and consider the adjoint d_L^* , where d_L is the derivative of the complex,

$$\dots \xrightarrow{d_L} \wedge^k \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \wedge^{k+1} \overline{L}_{\mathcal{J}} \xrightarrow{d_L} \dots$$

We also denote by G_L the Green operator of the Laplacian $\Delta_L := d_L d_L^* + d_L^* d_L$.

Let $\{\eta_i\}_{i=1}^m$ be a basis of the Harmonic forms $\mathbb{H}^2(\overline{L}) \cong H^2(\overline{L})$. As in (1.18) we also have the convergent series $\varepsilon(t)$ which is a unique solution of

$$\varepsilon(t) = \varepsilon_1(t) - \frac{1}{2} d_L^* G_L [\varepsilon(t), \varepsilon(t)]_S, \quad (7.1)$$

where $\varepsilon_1(t) = \sum_{i=1}^m \eta_i t_i$ and $t = (t_1, \dots, t_m) \in \mathbb{C}^m$. Note that $\varepsilon(t)$ is not a section with one variable but one with several variables $t = (t_1, \dots, t_m)$. The convergent series $\varepsilon(t)$ is determined by the first term $\varepsilon_1(t)$. The harmonic component of $[\varepsilon(t), \varepsilon(t)]_S$ is denoted by $H([\varepsilon(t), \varepsilon(t)]_S) \in H^3(\wedge^{\bullet} \overline{L})$. We define an analytic set A by

$$A = \{ t \in \mathbb{C}^m \mid |t| < \alpha, H([\varepsilon(t), \varepsilon(t)]_S) = 0 \}$$

where α is a sufficiently small constant.

PROPOSITION 7.1. *We have a family of generalized complex structures $\{\mathcal{J}_t\}$ which is parametrised by the analytic set A .*

Our proof is almost same as in the one of complex deformations and we use the similar notation as in [16].

PROOF. It suffices to show that for a fixed $\varepsilon_1(t)$, the $\varepsilon(t)$ in (7.1) satisfies the Maurer-Cartan equation if and only if $H([\varepsilon(t), \varepsilon(t)]_S) = 0$. If $\varepsilon(t)$ is a solution of the Maurer-Cartan equation,

$$d_L \varepsilon(t) + \frac{1}{2}[\varepsilon(t), \varepsilon(t)]_S = 0.$$

Then it follows that the harmonic part $H([\varepsilon(t), \varepsilon(t)]_S)$ vanishes. Conversely, we assume that $H([\varepsilon(t), \varepsilon(t)]_S) = 0$. Let $\Psi = d_L \varepsilon(t) + \frac{1}{2}[\varepsilon(t), \varepsilon(t)]_S \in \wedge^3 \overline{L}$. It follows from (7.1) that $d_L \varepsilon(t) = -\frac{1}{2}d_L d_L^* G_L [\varepsilon(t), \varepsilon(t)]_S$. Then applying the Hodge decomposition to $[\varepsilon(t), \varepsilon(t)]_S$, we have

$$2\Psi = -d_L d_L^* G_L [\varepsilon(t), \varepsilon(t)]_S + [\varepsilon(t), \varepsilon(t)]_S \quad (7.2)$$

$$= H([\varepsilon(t), \varepsilon(t)]_S) + d_L^* d_L G_L [\varepsilon(t), \varepsilon(t)]_S \quad (7.3)$$

$$= d_L^* d_L G_L [\varepsilon(t), \varepsilon(t)]_S \quad (7.4)$$

By using the proposition 1.2 and substituting $d_L \varepsilon(t) = \Psi - \frac{1}{2}[\varepsilon(t), \varepsilon(t)]_S$, we have

$$\Psi = d_L^* G_L [d_L \varepsilon(t), \varepsilon(t)]_S \quad (7.5)$$

$$= d_L G_L [\Psi, \varepsilon(t)]_S - d_L G_L \frac{1}{2} [\varepsilon(t), \varepsilon(t)]_S, \varepsilon(t)]_S \quad (7.6)$$

$$= d_L G_L [\Psi, \varepsilon(t)]_S. \quad (7.7)$$

We use the Sobolev norm $\|\cdot\|_s$ and the elliptic estimate,

$$\|\Psi\|_s < C_1 \|\Psi, \varepsilon(t)\|_{s-1} \quad (7.8)$$

$$< C_2 \|\Psi\|_s \|\varepsilon(t)\|_s, \quad (7.9)$$

where C_1, C_2 are positive constants. Thus for small t such that $C_2 \|\varepsilon(t)\|_s < 1$, it follows that $\Psi = 0$. Hence $\varepsilon(t)$ satisfies the Maurer-Cartan equation. \square

8 Appendix II

We will give a short explanation of the Schouten bracket and the proposition 1.2. Our definition of the Schouten bracket is called the Dervied bracket construction [19]. Let (X, \mathcal{J}) be a generalized complex manifold with th decomposition $(T \oplus T^*)^{\mathbb{C}} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$. We denote by $\wedge^{\bullet} \overline{L}_{\mathcal{J}}$ the skew-symmetric forms of $\overline{L}_{\mathcal{J}}$, which acts on differential forms

$\wedge^\bullet T^*$ by the spin representation. Let $K_{\mathcal{J}}$ be the canonical line bundle which is given by $K_{\mathcal{J}} = \{ \phi \in \wedge^\bullet T^* \mid L_{\mathcal{J}} \cdot \phi = 0 \}$. Then the space of differential forms is decomposed into irreducible representations: $\wedge^\bullet T^* = \bigoplus_{i=0}^{2n} U^{-n+p}$, where each component U^{-n+p} is given by $\wedge^p \overline{L}_{\mathcal{J}} \cdot K_{\mathcal{J}}$. For a section $\varepsilon \in \wedge^p \overline{L}_{\mathcal{J}}$, we denote by $|\varepsilon| := p$ the degree of ε . The exterior derivative d is decomposed into $d = \partial + \overline{\partial}$, where $\partial : U^{-n+p} \rightarrow U^{-n+p-1}$ and the complex conjugate $\overline{\partial} : U^{-n+p} \rightarrow U^{-n+p+1}$. We consider $\varepsilon \in \wedge^\bullet \overline{L}_{\mathcal{J}}$ is an operator from $K_{\mathcal{J}}$ to $U^{-n+|\varepsilon|}$ by the spin representation of $\wedge^\bullet \overline{L}_{\mathcal{J}}$ on $\wedge^\bullet T^*$. For $\varepsilon_1, \varepsilon_2 \in \wedge^\bullet \overline{L}_{\mathcal{J}}$, we define a graded bracket $[\cdot, \cdot]_G$ by $[\varepsilon_1, \varepsilon_2]_G = \varepsilon_1 \varepsilon_2 - (-1)^{|\varepsilon_1||\varepsilon_2|} \varepsilon_2 \varepsilon_1$. Let A be a differential operator acting on $\wedge^\bullet T^*$. If $A : U^{-n+i} \rightarrow U^{-n+i+a}$, for all i , A is an operator of degree $a = |A|$. For operators A, B of degree $|A|$ and $|B|$, we also have the graded bracket:

$$[A, B]_G := AB - (-1)^{|A||B|} BA.$$

The exterior derivative d admits the decomposition $d = \partial + \overline{\partial}$, where ∂ and $\overline{\partial}$ are operators of degree 1 and -1 respectively. Then d is an operator of odd degree and the graded commutator with $\varepsilon \in \wedge^\bullet \overline{L}_{\mathcal{J}}$ is given by

$$D\varepsilon := [d, \varepsilon]_G = d\varepsilon - (-1)^{|\varepsilon|} \varepsilon d.$$

Then we define Schouten bracket $[\varepsilon_1, \varepsilon_2]_S \in \wedge^{|\varepsilon_1|+|\varepsilon_2|-1} \overline{L}_{\mathcal{J}}$ by

$$[\varepsilon_1, \varepsilon_2]_S := [D\varepsilon_1, \varepsilon_2]_G = [[d, \varepsilon_1]_G, \varepsilon_2]_G = [[\partial, \varepsilon_1]_G, \varepsilon_2]_G, \quad (8.1)$$

where $[[\overline{\partial}, \varepsilon_1]_G, \varepsilon_2]_G = 0$ and $[\varepsilon_1, \varepsilon_2]_S \in \wedge^{|\varepsilon_1|+|\varepsilon_2|-1} \overline{L}_{\mathcal{J}}$. Let d_L be the derivative of the Lie algebroid $\overline{L}_{\mathcal{J}}$. Then we have

$$d_L \varepsilon = [\overline{\partial}, \varepsilon]_G \in \wedge^{p+1} \overline{L}_{\mathcal{J}} \quad (8.2)$$

(Refer to [9].) In fact, since we have $[\overline{\partial}, \varepsilon]_G f \phi = f [\overline{\partial}, \varepsilon]_G \phi + [\overline{\partial} f, \varepsilon]_G \phi = f [\overline{\partial}, \varepsilon]_G \phi$, the operator $[\overline{\partial}, \varepsilon]_G$ is regarded as an element of $\text{Hom}(K_{\mathcal{J}}, U^{-n+|\varepsilon|+1})$ and since $[[\overline{\partial}, \varepsilon]_G, \varepsilon_1]_G \phi = 0$ for $\phi \in K_{\mathcal{J}}$ and $\varepsilon_1 \in \wedge^\bullet \overline{L}_{\mathcal{J}}$, the commutator $[\overline{\partial}, \varepsilon]_G$ is also an element of $\wedge^{|\varepsilon|+1} \overline{L}_{\mathcal{J}}$ under the isomorphism $\wedge^p \overline{L}_{\mathcal{J}} \cong \text{Hom}(K_{\mathcal{J}}, U^{-n+p})$, which is given by the spin representation. Then we obtain an isomorphism between two complexes:

$$(\wedge^\bullet \overline{L}_{\mathcal{J}}, d_L) \cong (U^{-n+\bullet} \otimes K_{\mathcal{J}}^{-1}, [\overline{\partial}, \cdot]_G)$$

In fact we have

$$[\overline{\partial}, [\overline{\partial}, \varepsilon]_G]_G = [\overline{\partial}, (\overline{\partial} \varepsilon - (-1)^{|\varepsilon|} \varepsilon \overline{\partial})]_G \quad (8.3)$$

$$= \overline{\partial} \overline{\partial} \varepsilon - (-1)^{|\varepsilon|} \overline{\partial} \varepsilon \overline{\partial} - (-1)^{|\varepsilon|+1} \overline{\partial} \varepsilon \overline{\partial} - \varepsilon \overline{\partial} \overline{\partial} \quad (8.4)$$

$$= 0 \quad (8.5)$$

From now we identify $d_L \varepsilon$ with $[\overline{\partial}, \varepsilon]_G$.

We have the following relations of the graded bracket.

LEMMA 8.1.

$$[A, B]_G = -(-1)^{|A||B|}[B, A]_G, \quad (8.6)$$

the Jacobi identity of the graded bracket holds

$$[[A, B]_G, C]_G(-1)^{|A||C|} + [[B, C]_G, A]_G(-1)^{|B||A|} + [[C, A]_G, B]_G(-1)^{|C||B|} = 0 \quad (8.7)$$

PROOF. These follows from a direct calculations. \square

We also have the following three relations of the Schouten bracket,

LEMMA 8.2.

$$[\varepsilon_1, \varepsilon_2]_S = (-1)^{|\varepsilon_1||\varepsilon_2|}[\varepsilon_2, \varepsilon_1]_S$$

LEMMA 8.3.

$$d_L[\varepsilon_1, \varepsilon_2]_S = [d_L\varepsilon_1, \varepsilon_2]_S + (-1)^{|\varepsilon_1|}[\varepsilon_1, d_L\varepsilon_2]_S$$

LEMMA 8.4.

$$[[\varepsilon_1, \varepsilon_2]_S, \varepsilon_3]_S(-1)^{|\varepsilon_1||\varepsilon_3|} + [[\varepsilon_2, \varepsilon_3]_S, \varepsilon_1]_S(-1)^{|\varepsilon_2||\varepsilon_1|} + [[\varepsilon_3, \varepsilon_1]_S, \varepsilon_2]_S(-1)^{|\varepsilon_3||\varepsilon_2|} = 0$$

We shall show that every lemma follows from (8.2) and lemma 8.1.

Proof of lemma 8.2 for $\varepsilon_1, \varepsilon_2 \in \wedge^\bullet \overline{L}_{\mathcal{J}}$, we have

$$D[\varepsilon_1, \varepsilon_2]_G = [D\varepsilon_1, \varepsilon_2]_G + (-1)^{|\varepsilon_1|}[\varepsilon_1, D\varepsilon_2]_G = 0 \quad (8.8)$$

Since $[\varepsilon_1, \varepsilon_2]_G = 0$, we have $[D\varepsilon_1, \varepsilon_2]_G + (-1)^{|\varepsilon_1|}[\varepsilon_1, D\varepsilon_2]_G = 0$. Since $[\varepsilon_1, D\varepsilon_2]_G = -(-1)^{|\varepsilon_1|(|\varepsilon_2|+1)}[D\varepsilon_2, \varepsilon_1]_G$, we obtain

$$[D\varepsilon_1, \varepsilon_2]_G = (-1)^{|\varepsilon_1||\varepsilon_2|}[D\varepsilon_2, \varepsilon_1]_G$$

It implies that $[\varepsilon_1, \varepsilon_2]_S = (-1)^{|\varepsilon_1||\varepsilon_2|}[\varepsilon_2, \varepsilon_1]_S$. \square

Proof of lemma 8.3 From (8.1) we have

$$d_L[\varepsilon_1, \varepsilon_2]_S = [\overline{\partial}, [\varepsilon_1, \varepsilon_2]_S]_G = [\overline{\partial}, [D\varepsilon_1, \varepsilon_2]_G]_G \quad (8.9)$$

Applying the lemma 8.1, we have

$$(-1)^{|\varepsilon_2|}d_L[\varepsilon_1, \varepsilon_2]_S = [[D\varepsilon_1, \varepsilon_2]_G, \overline{\partial}]_G(-1)(-1)^{|\varepsilon_1|+|\varepsilon_2|-1}(-1)^{|\varepsilon_2|} \quad (8.10)$$

$$= [[D\varepsilon_1, \varepsilon_2]_G, \overline{\partial}]_G(-1)(-1)^{|\varepsilon_1|-1} \quad (8.11)$$

$$= [[\varepsilon_2, \overline{\partial}]_G, D\varepsilon_1]_G(-1)^{|\varepsilon_2|(|\varepsilon_1|-1)} \quad (8.12)$$

$$+ [[\overline{\partial}, D\varepsilon_1]_G, \varepsilon_2]_G(-1)^{|\varepsilon_2|} \quad (8.13)$$

From the lemma 8.1 , we also have

$$[\bar{\partial}, D\varepsilon_1]_G = [D\varepsilon_1, \bar{\partial}]_G (-1)(-1)^{|\varepsilon_1|-1} \quad (8.14)$$

$$= [\partial, \varepsilon_1]_G, \bar{\partial}]_G (-1)(-1)^{|\varepsilon_1|-1} \quad (8.15)$$

$$= [\varepsilon_1, \bar{\partial}]_G, \partial]_G (-1)^{|\varepsilon_1|-1} (-1)^{|\varepsilon_1|} \quad (8.16)$$

$$+ [\bar{\partial}, \partial]_G, \varepsilon_1]_G (-1)^{|\varepsilon_1|-1} (-1)^{|\varepsilon_1|} \quad (8.17)$$

Since $[\bar{\partial}, \partial]_G = \bar{\partial}\partial + \partial\bar{\partial} = 0$, we have

$$[\bar{\partial}, D\varepsilon_1]_G = [\varepsilon_1, \bar{\partial}]_G, \partial]_G (-1) \quad (8.18)$$

$$= [\bar{\partial}, \varepsilon_1]_G, \partial]_G (-1)^{|\varepsilon_1|} \quad (8.19)$$

$$= [\partial, [\bar{\partial}, \varepsilon_1]_G]_G (-1)^{|\varepsilon_1|} (-1)(-1)^{|\varepsilon_1|+1} \quad (8.20)$$

$$= Dd_L\varepsilon_1 \quad (8.21)$$

Substituting them, we obtain

$$(-1)^{|\varepsilon_2|} d_L[\varepsilon_1, \varepsilon_2]_S = [\bar{\partial}, \varepsilon_2]_G, D\varepsilon_1]_G (-1)^{|\varepsilon_2|} (-1)^{|\varepsilon_1|-1} (-1)(-1)^{|\varepsilon_2|} \quad (8.22)$$

$$+ [Dd_L\varepsilon_1, \varepsilon_2]_G (-1)^{|\varepsilon_2|} \quad (8.23)$$

$$(8.24)$$

Hence we have

$$d_L[\varepsilon_1, \varepsilon_2]_S = [D\varepsilon_1, [\bar{\partial}, \varepsilon_2]_G]_G (-1)^{|\varepsilon_2|} (-1)^{|\varepsilon_1|-1} (-1)(-1)^{(|\varepsilon_1|-1)(|\varepsilon_2|+1)} \quad (8.25)$$

$$+ [Dd_L\varepsilon_1, \varepsilon_2]_G \quad (8.26)$$

$$= [Dd_L\varepsilon_1, \varepsilon_2]_G + (-1)^{|\varepsilon_1|} [D\varepsilon_1, [\bar{\partial}, \varepsilon_2]_G]_G \quad (8.27)$$

$$= [d_L\varepsilon_1, \varepsilon_2]_S + (-1)^{|\varepsilon_1|} [\varepsilon_1, d_L\varepsilon_2]_S \quad (8.28)$$

□

Proof of lemma 8.4 For $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \wedge^\bullet \bar{L}_{\mathcal{J}}$, it follow from (8.8) that one have

$$\begin{aligned} [\varepsilon_1, \varepsilon_2]_S, \varepsilon_3]_S &= [D[D\varepsilon_1, \varepsilon_2]_G, \varepsilon_3]_G \\ &= [[D\varepsilon_1, D\varepsilon_2]_G, \varepsilon_3]_G (-1)^{(|\varepsilon_1|+1)} \end{aligned}$$

$$\begin{aligned} [\varepsilon_2, \varepsilon_3]_S, \varepsilon_1]_S &= [D[D\varepsilon_2, \varepsilon_3]_G, \varepsilon_1]_G \\ &= [D\varepsilon_2, \varepsilon_3]_G, D\varepsilon_1]_G (-1)^{(|\varepsilon_2|+|\varepsilon_3|)} \end{aligned}$$

$$\begin{aligned} [\varepsilon_3, \varepsilon_1]_S, \varepsilon_2]_S &= [D[D\varepsilon_3, \varepsilon_1]_G, \varepsilon_2]_G \\ &= [D[\varepsilon_3, D\varepsilon_1]_G, \varepsilon_2]_G (-1)^{(|\varepsilon_3|+1)} \\ &= [\varepsilon_3, D\varepsilon_1]_G, D\varepsilon_2]_G (-1)^{(|\varepsilon_3|+1)} (-1)^{(|\varepsilon_3|+|\varepsilon_1|)} \end{aligned}$$

Then we have three equations,

$$[[\varepsilon_1, \varepsilon_2]_S, \varepsilon_3]_S (-1)^{|\varepsilon_1||\varepsilon_3|} = [[D\varepsilon_1, D\varepsilon_2]_G, \varepsilon_3]_G (-1)^{(|\varepsilon_1|+1)|\varepsilon_3|} (-1)^{|\varepsilon_1||\varepsilon_3|} \quad (8.29)$$

$$[[\varepsilon_2, \varepsilon_3]_S, \varepsilon_1]_S (-1)^{|\varepsilon_2||\varepsilon_1|} = [[D\varepsilon_2, \varepsilon_3]_G, D\varepsilon_1]_G (-1)^{(|\varepsilon_2|+|\varepsilon_3|)|\varepsilon_1|} (-1)^{|\varepsilon_2||\varepsilon_1|} \quad (8.30)$$

$$[[\varepsilon_3, \varepsilon_1]_S, \varepsilon_2]_S (-1)^{|\varepsilon_3||\varepsilon_2|} = [[\varepsilon_3, D\varepsilon_1]_G, D\varepsilon_2]_G (-1)^{(|\varepsilon_3|+1)|\varepsilon_2|} (-1)^{(|\varepsilon_3|+|\varepsilon_1|)|\varepsilon_2|} (-1)^{|\varepsilon_3||\varepsilon_2|} \quad (8.31)$$

Multiplying $(-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|}$, we have

$$(-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|} [[\varepsilon_1, \varepsilon_2]_S, \varepsilon_3]_S (-1)^{|\varepsilon_1||\varepsilon_3|} = [[D\varepsilon_1, D\varepsilon_2]_G, \varepsilon_3]_G (-1)^{(|\varepsilon_1|+1)|\varepsilon_3|} (-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|} \quad (8.32)$$

$$(-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|} [[\varepsilon_2, \varepsilon_3]_S, \varepsilon_1]_S (-1)^{|\varepsilon_2||\varepsilon_1|} = [[D\varepsilon_2, \varepsilon_3]_G, D\varepsilon_1]_G (-1)^{(|\varepsilon_1|+1)(|\varepsilon_2|+1)|\varepsilon_1|} (-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|} \quad (8.33)$$

$$(-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|} [[\varepsilon_3, \varepsilon_1]_S, \varepsilon_2]_S (-1)^{|\varepsilon_3||\varepsilon_2|} = [[\varepsilon_3, D\varepsilon_1]_G, D\varepsilon_2]_G (-1)^{|\varepsilon_3|(|\varepsilon_2|+1)|\varepsilon_1|} (-1)^{(|\varepsilon_3|-|\varepsilon_1|-1)|\varepsilon_2|} \quad (8.34)$$

We apply the Jacobi identity of the graded bracket $[\cdot, \cdot]_G$

$$[[A, B]_G, C]_G (-1)^{|A||C|} + [[B, C]_G, A]_G (-1)^{|B||C|} + [[C, A]_G, B]_G (-1)^{|C||B|} = 0 \quad (8.35)$$

Then we have the Jacobi identity of the Schouten bracket

$$[[\varepsilon_1, \varepsilon_2]_S, \varepsilon_3]_S (-1)^{|\varepsilon_1||\varepsilon_3|} + [[\varepsilon_2, \varepsilon_3]_S, \varepsilon_1]_S (-1)^{|\varepsilon_2||\varepsilon_1|} + [[\varepsilon_3, \varepsilon_1]_S, \varepsilon_2]_S (-1)^{|\varepsilon_3||\varepsilon_2|} = 0.$$

□

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